

**D. K. Faddeev & I. S. Sominskii**

---

# **Problems in Higher Algebra**

**Translated by J. L. Brenner**

---



~~39~~  
295

## Problems in Higher Algebra

## A SERIES OF BOOKS IN MATHEMATICS

**R. A. Rosenbaum, *Editor***

- |  |  |
|--|--|
| <b>Daniel T. Finkbeiner, II</b>                                    | Introduction to Matrices and Linear Transformations                                      |
| <b>Henry L. Alder<br/>and Edward B. Roessler</b>                   | Introduction to Probability and Statistics (Third Edition)                               |
| <b>D. O. Shklarsky,<br/>N. N. Chentzov,<br/>and I. M. Yaglom</b>   | The USSR Olympiad Problem Book: Selected Problems and Theorems of Elementary Mathematics |
| <b>Sherman K. Stein</b>  | Mathematics: The Man-made Universe   |
| <b>Robert R. Stoll</b>   | Set Theory and Logic   |
| <b>J. L. Brenner</b>   | Problems in Differential Equations   |
| <b>A. I. Mal'cev</b>   | Foundations of Linear Algebra  |
| <b>D. K. Faddeev<br/>and V. N. Faddeeva</b>                        | Computational Methods of Linear Algebra  |
| <b>C. E. Springer</b>  | Geometry and Analysis of Projective Spaces   |
| <b>Jack R. Britton,<br/>R. Ben Kriegh,<br/>and Leon W. Rutland</b> | University Mathematics, I and II   |
| <b>D. K. Faddeev<br/>and I. S. Sominskii</b>                       | Problems in Higher Algebra   |

## GOLDEN GATE EDITIONS

- |   |   |
|---|---|
| <b>W. W. Sawyer</b>                           | A Concrete Approach to Abstract Algebra                 |
| <b>Leonard M. Blumenthal</b>                  | A Modern View of Geometry                               |
| <b>Robert R. Stoll</b>                        | Sets, Logic, and Axiomatic Theories                     |
| <b>A. O. Gelfond</b>                          | The Solution of Equations in Integers                   |
| <b>B. V. Gnedenko<br/>and A. Ya. Khinchin</b> | An Elementary Introduction to the Theory of Probability |
| <b>J. B. Roberts</b>                          | The Real Number System in an Algebraic Setting          |
| <b>Peter van de Kamp</b>                      | Elements of Astromechanics                              |

# Problems in Higher Algebra

*D. K. Faddeev, I. S. Sominskii*

*Translated by*

**J. L. BRENNER**

STANFORD RESEARCH INSTITUTE

**W. H. FREEMAN AND COMPANY**

**SAN FRANCISCO AND LONDON**





© Copyright 1965 by J. L. Brenner.

*The publisher reserves all rights to reproduce this book,  
in whole or in part, with the exception of the right to  
use short quotations for review of the book.  
Printed in the United States of America.*

*Library of Congress Catalog Card Number: 65-18946*

## PREFACE

Selections from this fine collection of problems have been quoted ever since its first appearance in 1954. It is time to make it more widely available to the western world.

The section on linear equations will be interesting because of its systematic coverage, and because it uses the vector-space of the  $n$ -tuples of coefficients to expound the principal results. Calculation of determinants has been lifted from the sterile to the fruitful by means of imaginative problems, many of them solved by use of mathematical induction. At the same time, there are plenty of simple problems, but written by a master who can make the material interesting.

The section on Sturm sequences includes applications to the classical polynomials--the polynomials of Legendre, of Hermite, of Laguerre; and to the truncated exponential series. A feature of the book is that every problem, or at the most every group of two or three problems, is independent, and sections of the book can be sampled or omitted at the user's discretion.

I have added a few problems to the section on matrices; I learned them at the 1964 Gatlinburg conference. I hope the mathematical public will find these and other additions attractive.

This book is recommended as an adjunct text, as a problem book, and for self study. It is a pleasure to acknowledge the splendid cooperation of the publisher, and the consummate skill of Shirley DeMeo, the typist-compositor.

Joel Brenner

Menlo Park, California

January, 1965

# TABLE OF CONTENTS

	<u>Page</u>
Preface	v
First Part PROBLEMS	
Chapter I. Complex Numbers . . . . .	1
1. Reckoning with Complex Numbers . . . . .	1
2. Complex Numbers in Polar Form . . . . .	4
3. Equations of the Third and Fourth Degree . . . . .	13
4. Roots of Unity . . . . .	15
Chapter II. Computation of Determinants . . . . .	22
1. Second and Third Order Determinants . . . . .	22
2. Permutations . . . . .	23
3. Definition of Determinants . . . . .	25
4. Fundamental Properties of Determinants . . . . .	28
5. Evaluation of Determinants . . . . .	32
6. Multiplication of Determinants . . . . .	53
7. Miscellaneous Problems . . . . .	60
Chapter III. Systems of Linear Equations . . . . .	66
1. Cramer's Rule . . . . .	66
2. The Rank of a Matrix . . . . .	70
3. Systems of Linear Forms . . . . .	73
4. Systems of Linear Equations . . . . .	75
Chapter IV. Matrices . . . . .	87
1. Operations with Square Matrices . . . . .	87
2. Rectangular Matrices. Certain Inequalities . . . . .	98

	<u>Page</u>
Chapter V. Polynomials and Rational Functions of a Single Indeterminate . . . . .	106
1. Properties of Polynomials. Taylor's Formula, Multiple Roots . . . . .	106
2. Proof of the Fundamental Theorem of Algebra and Related Questions . . . . .	112
3. Factoring into Linear Factors. Factoring into Factors that are Irreducible in the Field of Real Numbers. Formulas Connecting Coefficients and Roots . . . . .	114
4. Euclidean Algorithm . . . . .	122
5. The Interpolation Problem; Rational Functions . . . . .	126
6. Rational Roots of Polynomials. Reducibility and Irreducibility in the Field of Rational Numbers . . . . .	132
7. Bounds for the Roots of Polynomials . . . . .	137
8. Sturm's Theorem . . . . .	140
9. Miscellaneous Methods for the Location of Roots of Polynomials . . . . .	145
10. Approximate Calculation of the Roots of Polynomials . . . . .	151
Chapter VI. Symmetric Functions . . . . .	154
1. Symmetric Functions in Terms of the Elementary Symmetric Functions. Calculation of the Symmetric Functions of the Roots of an Algebraic Equation. . . . .	154
2. Power Sums . . . . .	162
3. Transformation of Equations . . . . .	165
4. Resultants and Discriminants . . . . .	168
5. Tschirnhaus Transformation and the Elimination of Irrationalities in the Denominator . . . . .	176
6. Polynomials that are Invariant under Even Permutation of the Variables. Polynomials that are Invariant under Circular Permutation of the Variables . . . . .	178

	<u>Page</u>
Chapter VII. Linear Algebra . . . . .	182
1. Subspaces and Linear Varieties.	
Coordinate Subspaces. . . . .	182
2. Elementary Geometry of n-Dimensional Euclidean Space . . . . .	186
3. Proper Values and Proper Vectors of a Matrix . . . . .	193
4. Quadratic Forms and Symmetric Matrices .	195
5. Linear Spaces. The Jordan Canonical Form	203

## Second Part

### HINTS

Chapter I. Complex Numbers . . . . .	211
Chapter II. Computation of Determinants . . . .	216
Chapter IV. Matrices . . . . .	230
Chapter V. Polynomials and Rational Functions of a Single Indeterminate . . . .	233
Chapter VI. Symmetric Functions . . . . .	241
Chapter VII. Linear Algebra . . . . .	244

## Third Part

### SOLUTIONS

Chapter I. Complex Numbers . . . . .	249
Chapter II. Computation of Determinants . . . .	274
Chapter III. Systems of Linear Equations . . . .	291
Chapter IV. Matrices . . . . .	303
Chapter V. Polynomials and Rational Functions of a Single Indeterminate . . . .	340
Chapter VI. Symmetric Functions . . . . .	415
Chapter VII. Linear Algebra . . . . .	453





## CHAPTER I - PROBLEMS

### COMPLEX NUMBERS

#### 1. RECKONING WITH COMPLEX NUMBERS

- 1 If  $x$  and  $y$  are real numbers, and if the relation
$$(1 + 2i)x + (3 - 5i)y = 1 - 3i,$$
is satisfied, find the values of  $x$  and  $y$ .
- 2 If  $x, y, z, t$  are real numbers, solve the following system of equations:
$$(1 + i)x + (1 + 2i)y + (1 + 3i)z + (1 + 4i)t = 1 + 5i,$$
$$(3 - i)x + (4 - 2i)y + (1 + i)z + 4it = 2 - i.$$
- 3 If  $n$  is a whole number, find the value of  $i^n$ .
- 4 Check the following identity
$$x^4 + 4 = (x - 1 - i)(x - 1 + i)(x + 1 + i)(x + 1 - i).$$
- 5 Perform the following computations:  
a)  $(1 + 2i)^6$ ; b)  $(2 + i)^7 + (2 - i)^7$ ; c)  $(1 + 2i)^5 - (1 - 2i)^5$ .
- 6 Under what conditions is the product of two complex numbers pure imaginary?

7 Calculate the following quantities:

$$\begin{array}{lll} \text{a) } \frac{1+i \operatorname{tg} \alpha}{1-i \operatorname{tg} \alpha}; & \text{b) } \frac{a+bi}{a-bi}; & \text{c) } \frac{(1+2i)^2-(1-i)^3}{(3+2i)^3-(2+i)^2}; \\ \text{d) } \frac{(1-i)^5-1}{(1+i)^5+1}; & \text{e) } \frac{(1+i)^9}{(1-i)^7}. \end{array}$$

8 If  $n$  is a rational integer, compute the value of

$$\text{the fraction } \frac{(1+i)^n}{(1-i)^{n-2}}.$$

9 Solve the following three systems of equations:

$$\begin{array}{ll} \text{a) } (3-i)x + (4+2i)y = 2+6i, & (4+2i)x - (2+3i)y = 5+4i; \\ \text{b) } (2+i)x + (2-i)y = 6, & (3+2i)x + (3-2i)y = 8; \\ \text{c) } x + yi - 2z = 10, & x - y + 2iz = 20, \quad ix + 3iy - (1+i)z = 30. \end{array}$$

10 Compute the following:

$$\text{a) } \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2; \quad \text{b) } \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^3.$$

\*11 If  $\omega$  stands for  $-\frac{1}{2} + \frac{i\sqrt{3}}{2}$ , compute the following:

$$\begin{array}{ll} \text{a) } (a+b\omega+c\omega^2)(a+b\omega^2+c\omega); & \\ \text{b) } (a+b)(a+b\omega)(a+b\omega^2); & \\ \text{c) } (a+b\omega+c\omega^2)^3 + (a+b\omega^2+c\omega)^3; & \\ \text{d) } (a\omega^2+b\omega)(b\omega^2+a\omega). & \end{array}$$

12 Find a complex number which is conjugate to

a) its square, b) its cube.

- \*13 Prove the following theorem. Suppose the numbers  $x_1, x_2, \dots, x_n$  are combined in any rational way (that is, using addition, subtraction, multiplication, or division) and the result of this combination is  $u$ . Then the result of carrying out the same rational operations on the conjugate complex numbers  $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$  will be the number  $\bar{u}$ ; that is a complex conjugate of  $u$ .
- 14 If  $x + yi = (s + ti)^n$ , then show that  $x^2 + y^2 = (s^2 + t^2)^n$ .
- 15 Compute the values of the following quantities:
- a)  $\sqrt{2i}$ ; b)  $\sqrt{-8i}$ ; c)  $\sqrt{3-4i}$ ; d)  $\sqrt{-15+8i}$ ;
  - e)  $\sqrt{-3-4i}$ ; f)  $\sqrt{-11+60i}$ ; g)  $\sqrt{-8+6i}$ ;
  - h)  $\sqrt{-8-6i}$ ; i)  $\sqrt{8-6i}$ ; j)  $\sqrt{8+6i}$ ; k)  $\sqrt{2-3i}$ ;
  - l)  $\sqrt{4+i} + \sqrt{4-i}$ ; m)  $\sqrt{1-i}\sqrt{3}$ ; n)  $\sqrt[4]{-1}$ ;
  - o)  $\sqrt[4]{2-i}\sqrt[4]{12}$ .
- 16 Given that  $\sqrt{a+bi} = \pm(\alpha + \beta i)$ , what are the values of  $\sqrt{-a-bi}$  ?
- 17 Solve the following equations:
- a)  $x^2 - (2+i)x + (-1+7i) = 0$ ;
  - b)  $x^2 - (3-2i)x + (5-5i) = 0$ ;
  - c)  $(2+i)x^2 - (5-i)x + (2-2i) = 0$ .

- \*18 Factor each of the following into factors with real coefficients and solve:

a)  $x^4 + 6x^3 + 9x^2 + 100 = 0$ ; b)  $x^4 + 2x^2 - 24x + 72 = 0$ .

- 19 Solve the equations:

a)  $x^4 - 3x^2 + 4 = 0$ ; b)  $x^4 - 30x^2 + 289 = 0$ .

- 20 If  $p^2/4 - q < 0$ , find a formula for solving the bi-quadratic equation  $x^4 + px^2 + q = 0$ .

## 2. COMPLEX NUMBERS IN POLAR FORM

- 21 In the Argand diagram, locate the following complex numbers:

1,  $-1$ ,  $-\sqrt{2}$ ,  $i$ ,  $-i$ ,  $i\sqrt{2}$ ,  $-1+i$ ,  $2-3i$ .

- 22 Find the polar form for the following complex numbers:

a) 1; b)  $-1$ ; c)  $i$ ; d)  $-i$ ; e)  $1+i$ ; f)  $-1+i$ ; g)  $-1-i$ ;  
h)  $1-i$ ; i)  $1+i\sqrt{3}$ ; j)  $-1+i\sqrt{3}$ ; k)  $-1-i\sqrt{3}$ ;  
l)  $1-i\sqrt{3}$ ; m)  $2i$ ; n)  $-3$ ; o)  $\sqrt{3}-i$ ; p)  $2+\sqrt{3}+i$ .

- 23 Use tables to calculate approximately the arguments of the following numbers and thus write them in polar form:

a)  $3+i$ ; b)  $4-i$ ; c)  $-2+i$ ; d)  $-1-2i$ .

- 24 Find the geometric locus of those points that satisfy the following conditions:
- a) the modulus of the points is 1;
  - b) the points all have argument  $\pi/6$ .
- 25 Find the geometric locus in the  $z$ -plane of those points that satisfy the following conditions:
- a)  $|z| < 2$ ; b)  $|z - i| \leq 1$ ; c)  $|z - 1 - i| < 1$ .
- 26 Solve the following equations:
- a)  $|x| - x = 1 + 2i$ ; b)  $|x| + x = 2 + i$ .
- \*27 Establish the following identity, valid for any two complex numbers  $x, y$ :
- $$|x + y|^2 + |x - y|^2 = 2(|x|^2 + |y|^2)$$
- Find a geometric interpretation of the above identity.
- \*28 Let  $z$  be any complex number except  $-1$ ,  $z \neq -1$ . Show that there is a real number  $t$ , such that  $z$  can be written in the form  $z = \frac{1 + ti}{1 - ti}$ .
- 29 Find conditions under which the modulus of the sum of two complex numbers is equal to the difference of the moduli of the individual terms.
- 30 Find conditions under which the modulus of the sum of two complex numbers is equal to the sum of the moduli of the individual terms.

- \*31 Let  $u = \sqrt{zz'}$ , where  $z$  and  $z'$  are two complex numbers. Establish the following identity:

$$|z| + |z'| = \left| \frac{z+z'}{2} - u \right| + \left| \frac{z+z'}{2} + u \right|.$$

- 32 If  $|z| < \frac{1}{2}$ , then show that the following relationship is satisfied:

$$|(1+i)z^3 + iz| < \frac{3}{4}.$$

- 33 Establish the following:

$$\begin{aligned} (1+i\sqrt{3})(1+i)(\cos \varphi + i \sin \varphi) = \\ = 2\sqrt{2} \left[ \cos \left( \frac{7\pi}{12} + \varphi \right) + i \sin \left( \frac{7\pi}{12} + \varphi \right) \right]. \end{aligned}$$

- 34 Simplify the fraction  $\frac{\cos \varphi + i \sin \varphi}{\cos \psi - i \sin \psi}$ .

- 35 Compute the following:  $\frac{(1-i\sqrt{3})(\cos \varphi + i \sin \varphi)}{2(1-i)(\cos \varphi - i \sin \varphi)}$ .

- 36 Compute the following quantities:

$$\begin{aligned} \text{a) } (1+i)^{25}, \quad \text{b) } \left( \frac{1+i\sqrt{3}}{1-i} \right)^{20}; \quad \text{c) } \left( 1 - \frac{\sqrt{3}-i}{2} \right)^{24}; \\ \text{d) } \frac{(-1+i\sqrt{3})^{15}}{(1-i)^{20}} + \frac{(-1-i\sqrt{3})^{15}}{(1+i)^{20}}. \end{aligned}$$

- 37 If  $n$  is a rational integer, show that the following relations hold:

$$\begin{aligned} \text{a) } (1+i)^n &= 2^{\frac{n}{2}} \left( \cos \frac{n\pi}{4} + i \sin \frac{n\pi}{4} \right); \\ \text{b) } (\sqrt{3}-i)^n &= 2^n \left( \cos \frac{n\pi}{6} - i \sin \frac{n\pi}{6} \right); \end{aligned}$$

\*38 Let  $\omega = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$ .

Calculate  $(1 + \omega)^n$ .

- 39 Compute the value of  $\omega_1^n + \omega_2^n$ , where  $n$  is a rational integer and  $\omega_1$ , and  $\omega_2$  are defined by

$$\omega_1 = -\frac{1}{2} + i \frac{\sqrt{3}}{2}, \quad \omega_2 = -\frac{1}{2} - i \frac{\sqrt{3}}{2}.$$

\*40 Compute  $(1 + \cos \alpha + i \sin \alpha)^n$ .

\*41 Given that  $z + \frac{1}{z} = 2 \cos \theta$ , show that

$$z^m + \frac{1}{z^m} = 2 \cos m\theta.$$

42 Establish the relation  $\left( \frac{1 + i \operatorname{tg} \alpha}{1 - i \operatorname{tg} \alpha} \right)^n = \frac{1 + i \operatorname{tg} n\alpha}{1 - i \operatorname{tg} n\alpha}$ .

43 Extract the following roots:

a)  $\sqrt[3]{i}$ ; b)  $\sqrt[3]{2-2i}$ ; c)  $\sqrt[4]{-4}$ ; d)  $\sqrt[6]{1}$ ; e)  $\sqrt[6]{-27}$ .

44 Compute the approximate numerical value of the following roots using trigonometric tables:

a)  $\sqrt[3]{2+i}$ ; b)  $\sqrt[3]{3-i}$ ; c)  $\sqrt[5]{2+3i}$ .

45 Compute the following:

a)  $\sqrt[6]{\frac{1-i}{\sqrt{3}+i}}$ ; b)  $\sqrt[8]{\frac{1+i}{\sqrt{3}-i}}$ ; c)  $\sqrt[6]{\frac{1-i}{1+i\sqrt{3}}}$ .

46 If it is known that  $\beta$  is one of the values of  $\sqrt[n]{\alpha}$ , find all values of  $\sqrt[n]{\alpha}$ .



47 Express the following in terms of  $\cos x$ ,  $\sin x$ :

a)  $\cos 5x$ ; b)  $\cos 8x$ ; c)  $\sin 6x$ ; d)  $\sin 7x$ .

48 Express  $\tan 6\varphi$  in terms of  $\tan \varphi$ .

49 Establish formulas that express  $\cos nx$ ,  $\sin nx$ , in terms of  $\cos x$ ,  $\sin x$ .

50 Express the following as trigonometric polynomials (a trigonometric polynomial is a sum of monomial terms, each term being a constant multiple of the sine or cosine of an integral multiple of  $x$ ):

a)  $\sin^3 x$ ; b)  $\sin^4 x$ ; c)  $\cos^5 x$ ; d)  $\cos^6 x$ .

\*51 Establish the following identities:

$$a) 2^{2m} \cos^{2m} x = 2 \sum_{k=0}^{m-1} C_k^{2m} \cos 2(m-k)x + C_m^{2m};$$

$$b) 2^{2m} \cos^{2m+1} x = \sum_{k=0}^m C_k^{2m+1} \cos(2m-2k+1)x$$

$$c) 2^{2m} \sin^{2m} x = 2 \sum_{k=0}^{m-1} (-1)^{m+k} C_k^{2m} \cos 2(m-k)x + C_m^{2m}$$

$$d) 2^{2m} \sin^{2m+1} x = \sum_{k=0}^m (-1)^{m+k} C_k^{2m+1} \sin(2m-2k+1)x.$$

\*52 Establish the following:  $2 \cos mx = (2 \cos x)^m - \frac{m}{1} (2 \cos x)^{m-2}$

$$+ \frac{m(m-3)}{1 \cdot 2} (2 \cos x)^{m-4} - \dots$$

$$+ (-1)^p \frac{m(m-p-1)(m-p-2) \dots (m-2p+1)}{p!} (2 \cos x)^{m-2p} + \dots$$

\*53 Express the fraction  $\frac{\sin mx}{\sin x}$  in terms of  $\cos x$ .

\*54 Collapse the following sums:

a)  $1 - C_2^n + C_4^n - C_6^n + \dots;$

b)  $C_1^n - C_3^n + C_5^n - C_7^n + \dots$

\*55 Establish the following identities:

a)  $1 + C_4^n + C_8^n + \dots = \frac{1}{2} \left( 2^{n-1} + 2^{\frac{n}{2}} \cos \frac{n\pi}{4} \right);$

b)  $C_1^n + C_5^n + C_9^n + \dots = \frac{1}{2} \left( 2^{n-1} + 2^{\frac{n}{2}} \sin \frac{n\pi}{4} \right);$

c)  $C_2^n + C_6^n + C_{10}^n + \dots = \frac{1}{2} \left( 2^{n-1} - 2^{\frac{n}{2}} \cos \frac{n\pi}{4} \right);$

d)  $C_3^n + C_7^n + C_{11}^n + \dots = \frac{1}{2} \left( 2^{n-1} - 2^{\frac{n}{2}} \sin \frac{n\pi}{4} \right).$

\*56 Compute the following sum:

$$C_1^n - \frac{1}{3} C_3^n + \frac{1}{9} C_5^n - \frac{1}{27} C_7^n + \dots$$

57 Let  $n$  be the largest integral multiple of 3 that does not exceed  $m$ . Establish the following relation

$$(x+a)^m + (x+a\omega)^m + (x+a\omega^2)^m \\ = 3x^m + 3C_3^m \omega^{m-3} a^3 + \dots + 3C_n^m \omega^{m-n} a^n,$$

where

$$\omega = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}.$$

58 Establish the following three relations:

a)  $1 + C_3^n + C_6^n + \dots = \frac{1}{3} \left( 2^n + 2 \cos \frac{n\pi}{3} \right);$

b)  $C_1^n + C_4^n + C_7^n + \dots = \frac{1}{3} \left( 2^n + 2 \cos \frac{(n-2)\pi}{3} \right);$

c)  $C_2^n + C_5^n + C_8^n + \dots = \frac{1}{3} \left( 2^n + 2 \cos \frac{(n-4)\pi}{3} \right).$

59 Compute the following sums:

- a)  $1 + a \cos \varphi + a^2 \cos 2\varphi + \dots + a^k \cos k\varphi$ ;
- b)  $\sin \varphi + a \sin (\varphi + h) + a^2 \sin (\varphi + 2h) + \dots + a^k \sin (\varphi + kh)$ ;
- c)  $\frac{1}{2} + \cos x + \cos 2x + \dots + \cos nx$ .

60 Establish the following:

$$\sin x + \sin 2x + \dots + \sin nx = \frac{\sin \frac{n+1}{2} x \cdot \sin \frac{nx}{2}}{\sin \frac{x}{2}}.$$

61 Compute the following limit:

$$\lim_{n \rightarrow \infty} \left( 1 + \frac{1}{2} \cos x + \frac{1}{4} \cos 2x + \dots + \frac{1}{2^n} \cos nx \right).$$

62 If  $n$  is a positive integer, and  $\theta$  is the angle that satisfies the relation  $\sin \frac{\theta}{2} = \frac{1}{2n}$ , show that the following relation is valid:

$$\cos \frac{\theta}{2} + \cos \frac{3\theta}{2} + \dots + \cos \frac{2n-1}{2} \theta = n \sin n\theta.$$

63 Establish each of the following:

- a)  $\cos \frac{\pi}{11} + \cos \frac{3\pi}{11} + \cos \frac{5\pi}{11} + \cos \frac{7\pi}{11} + \cos \frac{9\pi}{11} = \frac{1}{2}$ ;
- b)  $\cos \frac{2\pi}{11} + \cos \frac{4\pi}{11} + \cos \frac{6\pi}{11} + \cos \frac{8\pi}{11} + \cos \frac{10\pi}{11} = -\frac{1}{2}$ ;
- c)  $\cos \frac{\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} + \cos \frac{7\pi}{13} + \cos \frac{9\pi}{13} + \cos \frac{11\pi}{13} = \frac{1}{2}$ .

64 Calculate the following sums:

- a)  $\cos a - \cos (a + h) + \cos (a + 2h) - \dots$   
 $\dots + (-1)^{n-1} \cos [a + (n-1)h]$ ;
- b)  $\sin a - \sin (a + h) + \sin (a + 2h) - \dots$   
 $\dots + (-1)^{n-1} \sin [a + (n-1)h]$ .

- 65 If  $x$  has absolute value less than 1, show that the following series

$$\begin{aligned} \text{a) } & \cos \alpha + x \cos(\alpha + \beta) + x^2 \cos(\alpha + 2\beta) + \dots \\ & \dots + x^n \cos(\alpha + n\beta) + \dots, \\ \text{b) } & \sin \alpha + x \sin(\alpha + \beta) + x^2 \sin(\alpha + 2\beta) + \dots \\ & \dots + x^n \sin(\alpha + n\beta) + \dots \end{aligned}$$

respectively converge to the following values:

$$\frac{\cos \alpha - x \cos(\alpha - \beta)}{1 - 2x \cos \beta + x^2}, \quad \frac{\sin \alpha - x \sin(\alpha - \beta)}{1 - 2x \cos \beta + x^2}.$$

- 66 Calculate the following sums:

$$\begin{aligned} \text{a) } & \cos x + C_1^n \cos 2x + \dots + C_n^n \cos(n+1)x; \\ \text{b) } & \sin x + C_1^n \sin 2x + \dots + C_n^n \sin(n+1)x. \end{aligned}$$

- 67 Calculate the following sums:

$$\begin{aligned} \text{a) } & \cos x - C_1^n \cos 2x + C_2^n \cos 3x - \dots + (-1)^n C_n^n \cos(n+1)x; \\ \text{b) } & \sin x - C_1^n \sin 2x + C_2^n \sin 3x - \dots + (-1)^n C_n^n \sin(n+1)x. \end{aligned}$$

- \*68 Let the vectors defining the points 1,  $i$  on the Argand diagram be denoted respectively by  $\overrightarrow{OA_1}$  and  $\overrightarrow{OB}$  respectively. The perpendicular from  $O$  meets  $A_1B$  in  $A_2$ . The perpendicular from  $A_2$  meets  $OA_1$  in  $A_3$ , the perpendicular from  $A_3$  meets  $A_1A_2$  in  $A_4$  and so on. In general,  $A_nA_{n+1}$  is perpendicular to  $A_{n-2}A_{n-1}$ . Calculate the limit of the sum  $\overrightarrow{OA_1} + \overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \dots$

\*69 Compute the following sum

$$\sin^2 x + \sin^2 3x + \dots + \sin^2 (2n-1)x.$$

70 Establish the following identities:

$$a) \cos^2 x + \cos^2 2x + \dots + \cos^2 nx = \frac{n}{2} + \frac{\cos(n+1)x \sin nx}{2 \sin x};$$

$$b) \sin^2 x + \sin^2 2x + \dots + \sin^2 nx = \frac{n}{2} - \frac{\cos(n+1)x \sin nx}{2 \sin x}.$$

\*71 Compute the sums:

$$a) \cos^3 x + \cos^3 2x + \dots + \cos^3 nx;$$

$$b) \sin^3 x + \sin^3 2x + \dots + \sin^3 nx.$$

\*72 Compute the sums:

$$a) \cos x + 2 \cos 2x + 3 \cos 3x + \dots + n \cos nx;$$

$$b) \sin x + 2 \sin 2x + 3 \sin 3x + \dots + n \sin nx.$$

73 Let  $\alpha = a + bi$ . Compute the limit

$$\lim_{n \rightarrow \infty} \left(1 + \frac{\alpha}{n}\right)^n$$

74 Definition:  $e^\alpha = \lim_{n \rightarrow \infty} \left(1 + \frac{\alpha}{n}\right)^n$ .

Establish the following:

$$a) e^{2\pi i} = 1; \quad b) e^{\pi i} = -1;$$

$$c) e^{\alpha+\beta} = e^\alpha \cdot e^\beta; \quad d) (e^\alpha)^k = e^{\alpha k}$$

where  $k$  is an integer.

## 3. EQUATIONS OF THE THIRD AND FOURTH DEGREE

75 Use Cardan's formula to solve the following equations:

a)  $x^3 - 6x + 9 = 0$ ;

b)  $x^3 + 12x + 63 = 0$ ;

c)  $x^3 + 9x^2 + 18x + 28 = 0$ ;

d)  $x^3 + 6x^2 + 30x + 25 = 0$ ;

e)  $x^3 - 6x + 4 = 0$ ;

f)  $x^3 + 6x + 2 = 0$ ;

g)  $x^3 + 18x + 15 = 0$ ;

h)  $x^3 - 3x^2 - 3x + 11 = 0$ ;

i)  $x^3 + 3x^2 - 6x + 4 = 0$ ;

j)  $x^3 + 9x - 26 = 0$ ;

k)  $x^3 + 24x - 56 = 0$ ;

l)  $x^3 + 45x - 98 = 0$ ;

m)  $x^3 + 3x^2 - 3x - 1 = 0$ ;

n)  $x^3 - 6x^2 + 57x - 196 = 0$ ;

o)  $x^3 + 3x - 2i = 0$ ;

p)  $x^3 - 6ix + 4(1 - i) = 0$ ;

q)  $x^3 - 3abx + a^3 + b^3 = 0$ ;

r)  $x^3 - 3abfgx + f^2ga^3 + fg^2b^3 = 0$ ;

s)  $x^3 - 4x - 1 = 0$ ;

t)  $x^3 - 4x + 2 = 0$ .

\*76 Use Cardan's formula to establish the following relation, where  $x_1, x_2, x_3$  are the roots of the equation  $x^3 + px + q = 0$ .

$$(x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2 = -4p^3 - 27q^2,$$

(The expression  $-4p^3 - 27q^2$  is called the discriminant of the equation  $x^3 + px + q = 0$ .)

\*77 Solve the following equation:

$$(x^3 - 3qx + p^3 - 3pq)^2 - 4(px + q)^3 = 0.$$

\*78 Find a formula for solving the equation

$$x^5 - 5ax^3 + 5a^2x - 2b = 0.$$

79 Solve each of the following.

- a)  $x^4 - 2x^3 + 2x^2 + 4x - 8 = 0$ ; b)  $x^4 + 2x^3 - 2x^2 + 6x - 15 = 0$ ;  
 c)  $x^4 - x^3 - x^2 + 2x - 2 = 0$ ; d)  $x^4 - 4x^3 + 3x^2 + 2x - 1 = 0$ ;  
 e)  $x^4 - 3x^3 + x^2 + 4x - 6 = 0$ ; f)  $x^4 - 6x^3 + 6x^2 + 27x - 56 = 0$ ;  
 g)  $x^4 - 2x^3 + 4x^2 - 2x + 3 = 0$ ; h)  $x^4 - x^3 - 3x^2 + 5x - 10 = 0$ ;  
 i)  $x^4 + 2x^3 + 8x^2 + 2x + 7 = 0$ ; j)  $x^4 + 6x^3 + 6x^2 - 8 = 0$ ;  
 k)  $x^4 - 6x^3 + 10x^2 - 2x - 3 = 0$ ; l)  $x^4 - 2x^3 + 4x^2 + 2x - 5 = 0$ ;  
 m)  $x^4 - x^3 - 3x^2 + x + 1 = 0$ ; n)  $x^4 - x^3 - 4x^2 + 4x + 1 = 0$ ;  
 o)  $x^4 - 2x^3 + x^2 + 2x - 1 = 0$ ; p)  $x^4 - 4x^3 - 20x^2 - 8x + 4 = 0$ ;  
 q)  $x^4 - 2x^3 + 3x^2 - 2x - 2 = 0$ ; r)  $x^4 - x^3 + 2x - 1 = 0$ ;  
 s)  $4x^4 - 4x^3 + 3x^2 - 2x + 1 = 0$ ; t)  $4x^4 - 4x^3 - 6x^2 + 2x + 1 = 0$ .

80 Ferrari's algorithm for solving the fourth degree equation  $x^4 + ax^3 + bx^2 + cx + d = 0$  consists in writing the left member in the form

$$\left(x^2 + \frac{a}{2}x + \frac{\lambda}{2}\right)^2 - \left[\left(\frac{a^2}{4} + \lambda - b\right)x^2 + \left(\frac{a\lambda}{2} - c\right)x + \left(\frac{\lambda^2}{4} - d\right)\right]$$

and noting that if  $\lambda$  is properly chosen the square bracket will be a perfect square and thus the expression can be factored. A necessary condition that this occur is

$$\left(\frac{a\lambda}{2} - c\right)^2 - 4\left(\frac{a^2}{4} + \lambda - b\right)\left(\frac{\lambda^2}{4} - d\right) = 0,$$

Thus  $\lambda$  must be the solution of a "resolvent"



equation of the third degree. Once  $\lambda$  is determined the left member of the original equation can be factored.

Show that conversely the value of  $\lambda$  can be expressed in terms of the roots of the original fourth degree equation.

#### 4. ROOTS OF UNITY

- 81 Find the roots of unity of the following orders:  
a) 2; b) 3; c) 4; d) 6; e) 8; f) 12; g) 24.
- 82 Find primitive roots of unity of the following orders:  
a) 2; b) 3; c) 4; d) 6; e) 8; f) 12; g) 24.
- 83 To what exponent do the following quantities belong:  
a)  $z_k = \cos \frac{2k\pi}{180} + i \sin \frac{2k\pi}{180}$  for  $k = 27, 99, 137$ ;  
b)  $z_k = \cos \frac{2k\pi}{144} + i \sin \frac{2k\pi}{144}$  for  $k = 10, 35, 60$ ?
- 84 Find those 28-th units of unity that belong to exponent 7.
- 85 Find the exponent to which the various roots of unity belong, the degrees of the roots being  
a) 16; b) 20; c) 24.

- 86 Find the cyclotomic polynomial  $X_n(x)$  for the following values of  $n$ :

a) 1; b) 2; c) 3; d) 4; e) 5; f) 6;  
 g) 7; h) 8; i) 9; j) 10; k) 11; l) 12;  
 m) 15; n) 105.

- \*87 Let  $\epsilon$  be a primitive  $2n$ -th root of 1. Calculate the following sums:

$$1 + \epsilon + \epsilon^2 + \dots + \epsilon^{n-1}.$$

- \*88 Calculate the sum of all  $n$ -th roots of unity.

- \*89 Calculate the sum of the  $k$ -th powers of all  $n$ -th roots of unity.

- 90 Calculate the sum of the  $m$  quantities  $(x + a)^m$  where  $a$  runs through the  $m$ -th roots of unity.

- \*91 Let  $\epsilon$  be an  $n$ -th root of unity. Calculate

$$1 + 2\epsilon + 3\epsilon^2 + \dots + n\epsilon^{n-1}$$

- \*92 Let  $\epsilon$  be an  $n$ -th root of unity. Calculate

$$1 + 4\epsilon + 9\epsilon^2 + \dots + n^2\epsilon^{n-1}$$

- 93 Calculate the following sums:

$$a) \cos \frac{2\pi}{n} + 2 \cos \frac{4\pi}{n} + \dots + (n-1) \cos \frac{2(n-1)\pi}{n};$$

$$b) \sin \frac{2\pi}{n} + 2 \sin \frac{4\pi}{n} + \dots + (n-1) \sin \frac{2(n-1)\pi}{n}.$$

- \*94 Compute the sum of the primitive roots of unity of respective degrees: a) 15; b) 24; c) 30.

- 95 Calculate the 5-th roots of unity, that is the solutions of the equation  $x^5 - 1 = 0$ , in both algebraic and polar form.
- 96 Use the result of problem 95 to compute  $\sin 18^\circ$ ,  $\cos 18^\circ$ .
- \*97 Set up an algebraic equation the roots of which are the length of the sides of a regular 14-gon with circumradius 1.
- \*98 Factor the polynomial  $x^n - 1$  into polynomials of first and second degree with real coefficients.
- \*99 Use the results of problem 98 to establish the following formulas:

$$\text{a) } \sin \frac{\pi}{2m} \cdot \sin \frac{2\pi}{2m} \dots \sin \frac{(m-1)\pi}{2m} = \frac{\sqrt{m}}{2^{m-1}};$$

$$\text{b) } \sin \frac{\pi}{2m+1} \cdot \sin \frac{2\pi}{2m+1} \dots \sin \frac{m\pi}{2m+1} = \frac{\sqrt{2m+1}}{2^m}.$$

- \*100 Establish the following  $\prod_{k=0}^{n-1} (a + b\varepsilon_k) = a^n + (-1)^{n-1}b^n$ ,

where

$$\varepsilon_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}.$$

- \*101 Establish the following

$$\prod_{k=0}^{n-1} (\varepsilon_k^2 - 2\varepsilon_k \cos \theta + 1) = 2(1 - \cos n\theta),$$

where

$$\varepsilon_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}.$$

102 Establish the following

$$\prod_{k=0}^{n-1} \frac{(t + \varepsilon_k)^n - 1}{t} = \prod_{k=1}^{n-1} [t^n - (\varepsilon_k - 1)^n],$$

where  $\varepsilon_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}.$

\*103 Find all complex numbers that satisfy the relation  $\bar{x} = x^{n-1}$ , where  $\bar{x}$  is the complex conjugate of  $x$ .

104 Show that the roots of the equation  $\lambda(z - a)^n + \mu(z - b)^n = 0$ , where  $\lambda, \mu, a, b$  are complex, lie on a circle which can in special cases reduce to a line segment ( $n$  is a rational integer).

\*105 Solve each of the following:

a)  $(x + 1)^m - (x - 1)^m = 0$ ; b)  $(x + i)^m - (x - i)^m = 0$ ;  
c)  $x^n - nax^{n-1} - C_2^n a^2 x^{n-2} - \dots - a^n = 0.$

106 Show that if  $A$  is a complex number of modulus 1, all the roots of the equation

$$\left( \frac{1 + ix}{1 - ix} \right)^m = A$$

are real and distinct.

\*107 Solve the equation

$$\cos \varphi + C_1^n \cos(\varphi + \alpha) x + C_2^n \cos(\varphi + 2\alpha) x^2 + \dots \\ \dots + C_n^n \cos(\varphi + n\alpha) x^n = 0.$$

Prove the following theorems:

- 108 The product of an  $a$ -th root of unity by a  $b$ -th root of unity is an  $ab$ -th root of unity.
- 109 Let  $a, b$  be relatively prime integers. Then  $x^a - 1, x^b - 1$  have only the single trivial common root.
- 110 Let  $a, b$  be relatively prime integers. Then all the  $ab$ -th roots of unity can be obtained as products of  $a$ -th roots of unity and  $b$ -th roots of unity.
- 111 Let  $a, b$  be relatively prime. Then the product of a primitive  $a$ -th root of unity by a primitive  $b$ -th root of unity is a primitive  $ab$ -th root of unity, and conversely.
- 112 Let  $\varphi(n)$  be the number of primitive  $n$ -th roots of unity. Show that if  $a, b$  are relatively prime the relation  $\varphi(ab) = \varphi(a) \varphi(b)$  holds.
- \*113 Let  $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ , where  $p_1, p_2, \dots, p_k$  are distinct prime numbers (canonical decomposition of  $n$ ). Show that
- $$\varphi(n) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \dots \left(1 - \frac{1}{p_k}\right).$$
- 114 Show that the number of primitive  $n$ -th roots of unity,  $n > 2$ , is even.
- 115 If  $p$  is a prime number, calculate the polynomial  $X_p(x)$ , the cyclotomic polynomial.

\*116 If  $p$  is a prime number, calculate the cyclotomic polynomial  $X_{p^m}(x)$ .

\*117 If  $n$  is an odd number,  $n > 1$ , show that  $X_{2n}(x) = X_n(-x)$ .

118 If  $d$  is a product of prime numbers each of which divides  $n$ , then every primitive  $nd$ -th root of 1 is a  $d$ -th root of a primitive  $n$ -th root of 1, and conversely.

\*119 Let the canonical decomposition of  $n$  be  $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ , where  $p_1, p_2, \dots, p_k$  are distinct prime numbers. Then we have  $X_n(x) = X_{n'}(x^{n''})$ , where

$$n' = p_1 p_2 \dots p_k; \quad n'' = \frac{n}{n'}.$$

\*120 Let  $\mu(n)$  be the sum of the primitive  $n$ -th roots of 1. Show that  $\mu(n)$  is 0 whenever  $n$  is divisible by the square of a prime number;  $\mu(n)$  is 1 if  $n$  is the product of an even number of distinct prime numbers;  $\mu(n) = -1$  if  $n$  is the product of an odd number of distinct prime numbers.

121 Let  $d$  run through all divisors of  $n$ ,  $n \neq 1$ . Show that  $\sum \mu(d) = 0$ .

\*122 Establish the relation

$$X_n(x) = \prod (x^d - 1)^{\mu\left(\frac{n}{d}\right)}$$

where  $d$  runs through the divisors of  $n$ .

\*123 Calculate  $X_n(1)$ .

- \*124 Calculate  $X_n(-1)$ .
- \*125 Find the value of the sum of products of all possible pairs of primitive  $n$ -th roots of 1.
- \*126 Find  $|S|$  where  $S = 1 + \epsilon + \epsilon^4 + \epsilon^9 + \dots + \epsilon^{(n-1)^2}$  and where  $\epsilon$  is a primitive  $n$ -th root of 1.



## CHAPTER II - PROBLEMS

### COMPUTATION OF DETERMINANTS

#### 1. SECOND AND THIRD ORDER DETERMINANTS

Calculate the determinants of the following matrices:

127

$$\begin{aligned}
 &\text{a) } \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}; \text{ b) } \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}; \text{ c) } \begin{bmatrix} \sin \alpha & \cos \alpha \\ -\cos \alpha & \sin \alpha \end{bmatrix}; \\
 &\text{d) } \begin{bmatrix} a & c+di \\ c-di & b \end{bmatrix}; \text{ e) } \begin{bmatrix} \alpha+\beta i & \gamma+\delta i \\ \gamma-\delta i & \alpha-\beta i \end{bmatrix}; \text{ f) } \begin{bmatrix} \sin \alpha & \cos \alpha \\ \sin \beta & \cos \beta \end{bmatrix}; \\
 &\text{g) } \begin{bmatrix} \cos \alpha & \sin \alpha \\ \sin \beta & \cos \beta \end{bmatrix}; \text{ h) } \begin{bmatrix} \operatorname{tg} \alpha & -1 \\ 1 & \operatorname{tg} \alpha \end{bmatrix}; \text{ i) } \begin{bmatrix} 1+\sqrt{2} & 2-\sqrt{3} \\ 2+\sqrt{3} & 1-\sqrt{2} \end{bmatrix}; \\
 &\text{j) } \begin{bmatrix} 1 & \lg_b a \\ \lg_a b & 1 \end{bmatrix}; \text{ k) } \begin{bmatrix} a+b & b+d \\ a+c & c+d \end{bmatrix}; \text{ l) } \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix}; \\
 &\text{m) } \begin{bmatrix} x-1 & 1 \\ x^3 & x^2+x+1 \end{bmatrix}; \text{ n) } \begin{bmatrix} \omega & \omega \\ -1 & \omega \end{bmatrix},
 \end{aligned}$$

where

$$\omega = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3};$$

$$\text{where o) } \begin{bmatrix} \varepsilon & 1 \\ -1 & \varepsilon \end{bmatrix},$$

$$\varepsilon = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}.$$

128

$$\begin{aligned}
 &\text{a) } \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}; \text{ b) } \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}; \text{ c) } \begin{bmatrix} a & a & a \\ -a & a & x \\ -a & -a & x \end{bmatrix}; \\
 &\text{d) } \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}; \text{ e) } \begin{bmatrix} 1 & i & 1+i \\ -i & 1 & 0 \\ 1-i & 0 & 1 \end{bmatrix};
 \end{aligned}$$

$$f) \begin{bmatrix} 1 & \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} & \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \\ \cos \frac{\pi}{3} - i \sin \frac{\pi}{3} & 1 & \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \\ \cos \frac{\pi}{4} - i \sin \frac{\pi}{4} & \cos \frac{2\pi}{3} - i \sin \frac{2\pi}{3} & 1 \end{bmatrix};$$

$$g) \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}, \text{ where } \omega = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3};$$

$$h) \begin{bmatrix} 1 & 1 & \omega \\ 1 & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{bmatrix}, \text{ where } \omega = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}.$$

## 2. PERMUTATIONS

- 129 Write out the transpositions whose product changes the permutation 1, 2, 4, 3, 5 into the permutation 2, 5, 3, 4, 1.
- 130 How many inversions from natural order 1, 2, 3, 4, 5, 6, 7, 8, 9 are there in the following sequences:
- a) 1, 3, 4, 7, 8, 2, 6, 9, 5;
  - b) 2, 1, 7, 9, 8, 6, 3, 5, 4;
  - c) 9, 8, 7, 6, 5, 4, 3, 2, 1.

- 131 The permutations below are rearrangements of the natural order  $1, 2, 3, 4, 5, 6, 7, 8, 9$ . Determine  $i$  and  $k$  so that
- a) the permutation  $1, 2, 7, 4, i, 5, 6, k, 9$  is an even permutation;
  - b) the permutation  $1, i, 2, 5, k, 4, 8, 9, 7$  is an odd permutation.
- \*132 How many inversions from the natural order  $1, 2, \dots, n$  are there in the permutation  $n, n-1, \dots, 2, 1$ .
- \*133 Suppose the permutation  $\alpha_1, \alpha_2, \dots, \alpha_n$  has  $I$  inversions. How many inversions are there in the permutation  $\alpha_n, \alpha_{n-1}, \dots, \alpha_2, \alpha_1$ ?
- 134 How many permutations from the natural order  $1, 2, \dots, 2n$  are there in the permutations
- a)  $1, 3, 5, 7, \dots, 2n-1, 2, 4, 6, \dots, 2n$ ;
  - b)  $2, 4, 6, 8, \dots, 2n, 1, 3, 5, \dots, 2n-1$ .
- 135 How many inversions from the natural order  $1, 2, 3, \dots, 3n$  are there in the permutations
- a)  $3, 6, 9, \dots, 3n, 1, 4, 7, \dots, 3n-2, 2, 5, \dots, 3n-1$ ;
  - b)  $1, 4, 7, \dots, 3n-2, 2, 5, \dots, 3n-1, 3, 6, \dots, 3n$ ?
- 136 If we think of  $1, 2, \dots, n$  as the original order and suppose that the order  $a_1, a_2, \dots, a_n$  has  $I$  inversions, show that by changing the roles of these two permutations the number of inversions in going from the second order as the natural order to the first as an inverted order is the same.

- 137 Find whether the permutation t, h, r, m, i, a, g, o, l, is an even or odd permutation, assuming that the natural order of the letters is that in the word:

a) logarithm;

b) algorithm.

### 3. DEFINITION OF DETERMINANTS

- 138 If a determinant is defined as the sum of  $n!$  terms, in the usual way, what sign must prefix each of the following terms in the expansion of the determinant of the 6-th order matrix  $[a_{ij}]$ :

a)  $a_{23}a_{31}a_{42}a_{56}a_{14}a_{65}$ ; b)  $a_{32}a_{43}a_{14}a_{51}a_{66}a_{25}$ ?

- 139 Which of the following products actually appears in the expansion of the determinant of the 5-th order matrix  $[a_{ij}]$ :

a)  $a_{13}a_{24}a_{23}a_{41}a_{55}$ ; b)  $a_{21}a_{13}a_{34}a_{55}a_{42}$ ?

- 140 In the expansion of the determinant of the 5-th order matrix  $[a_{ij}]$   $a_{1i}a_{32}a_{4k}a_{25}a_{53}$  appears with a plus sign. Find i and k.

- 141 Write down all the terms that appear prefixed by minus sign, in the expansion of the determinant of the 4-th order matrix  $[a_{ij}]$  and which contain the factor  $a_{23}$ .

- 142 Which terms of the form  $a_{14}a_{23}a_{3\alpha_3}a_{4\alpha_4}a_{5\alpha_5}$  appear in the expansion of the determinant of the 5-th order matrix  $[a_{ij}]$ ? The sum of these terms naturally has the common factor  $a_{14}a_{23}$ . What is a convenient way of writing the other factor?
- 143 One of the terms in the expansion of the determinant of the  $n$ -th order matrix  $[a_{ij}]$  is the product of the elements in the principal diagonal. What sign is prefixed to this term in the expansion of the determinant?
- 144 One of the terms in the expansion of the determinant of the  $n$ -th order matrix  $[a_{ij}]$  is the product of the elements in the secondary diagonal. What sign is prefixed to this term in the expansion of the determinant?
- 145 Use only the definition of the determinant of the matrix to show that the determinant of the following matrix is 0:

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ a_1 & a_2 & 0 & 0 & 0 \\ b_1 & b_2 & 0 & 0 & 0 \\ c_1 & c_2 & 0 & 0 & 0 \end{bmatrix}$$

- 146 Use the definition of the determinant of the matrix to calculate the coefficients of  $x^4$  and  $x^3$  in the

expansion of the determinant:

$$f(x) = \det \begin{bmatrix} 2x & x & 1 & 2 \\ 1 & x & 1 & -1 \\ 3 & 2 & x & 1 \\ 1 & 1 & 1 & x \end{bmatrix}.$$

147 Evaluate the following determinants:

$$\text{a) } \det \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & n \end{bmatrix}; \quad \text{b) } \det \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix};$$

$$\text{c) } \det \begin{bmatrix} 1 & a & a & \dots & a \\ 0 & 2 & a & \dots & a \\ 0 & 0 & 3 & \dots & a \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & n \end{bmatrix}.$$

Note: In all succeeding exercises assume that the order of the matrix is  $n$  unless otherwise given.

148 Define  $F(x) = x(x-1)(x-2)\dots(x-n+1)$ .

Calculate the values:

$$\text{a) } \det \begin{bmatrix} F(0) & F(1) & F(2) & \dots & F(n) \\ F(1) & F(2) & F(3) & \dots & F(n+1) \\ \dots & \dots & \dots & \dots & \dots \\ F(n) & F(n+1) & F(n+2) & \dots & F(2n) \end{bmatrix};$$

$$\text{b) } \det \begin{bmatrix} F(a) & F'(a) & F''(a) & \dots & F^{(n)}(a) \\ F'(a) & F''(a) & F'''(a) & \dots & F^{(n+1)}(a) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F^{(n)}(a) & F^{(n+1)}(a) & F^{(n+2)}(a) & \dots & F^{(2n)}(a) \end{bmatrix}.$$

#### 4. FUNDAMENTAL PROPERTIES OF DETERMINANTS

\*149 Let an  $n$ -th order matrix be such that the  $ik$  element  $a_{ik}$  is the complex conjugate of  $ki$  element  $a_{ki}$ . Show that the determinant of the matrix is real.

\*150 Let the matrix  $[a_{ij}]$  have odd order, and suppose the sum of the symmetrically situated elements is 0:

$$a_{ik} + a_{ki} = 0.$$

Show that the determinant of the matrix is 0 (the determinant of a skew-symmetric matrix of odd order is 0).

151 Suppose the determinant of the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

is  $\Delta$ . Find the value of the determinant of the matrix

$$\begin{bmatrix} a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \\ a_{11} & a_{12} & \dots & a_{1n} \end{bmatrix}$$

152 How is the value of the determinant of a matrix changed if the columns are written in inverse order?

\*153 Calculate the value of the sum of determinants

$$\sum \det \begin{bmatrix} a_{1\alpha_1} & a_{1\alpha_2} & \dots & a_{1\alpha_n} \\ a_{2\alpha_1} & a_{2\alpha_2} & \dots & a_{2\alpha_n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n\alpha_1} & a_{n\alpha_2} & \dots & a_{n\alpha_n} \end{bmatrix},$$

the sum being extended over all possible permutations  $\alpha_1, \alpha_2, \dots, \alpha_n$ .

\*154 Let  $a_1, a_2, \dots, a_{n-1}$  be different numbers. Solve the following equations:

$$\text{a) } \det \begin{bmatrix} 1 & x & x^2 & \dots & x^{n-1} \\ 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & a_{n-1} & a_{n-1}^2 & \dots & a_{n-1}^{n-1} \end{bmatrix} = 0,$$



$$\text{b) det} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 1-x & 1 & \dots & 1 \\ 1 & 1 & 2-x & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & (n-1)-x \end{bmatrix} = 0;$$

$$\text{c) det} \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_1 & a_1 + a_2 - x & a_3 & \dots & a_n \\ a_1 & a_2 & a_2 + a_3 - x & \dots & a_n \\ \dots & \dots & \dots & \dots & \dots \\ a_1 & a_2 & a_3 & \dots & a_{n-1} + a_n - x \end{bmatrix} = 0.$$

- \*155 Noting that each of the numbers 204, 527, and 255 is divisible by 17, show that the determinant of the matrix

$$\begin{bmatrix} 2 & 0 & 4 \\ 5 & 2 & 7 \\ 2 & 5 & 5 \end{bmatrix}$$

is divisible by 17.

- \*156 Calculate the determinant of the matrix

$$\begin{bmatrix} \alpha^2 & (\alpha+1)^2 & (\alpha+2)^2 & (\alpha+3)^2 \\ \beta^2 & (\beta+1)^2 & (\beta+2)^2 & (\beta+3)^2 \\ \gamma^2 & (\gamma+1)^2 & (\gamma+2)^2 & (\gamma+3)^2 \\ \delta^2 & (\delta+1)^2 & (\delta+2)^2 & (\delta+3)^2 \end{bmatrix}.$$

- 157 Establish the following relation between the two determinants shown

$$\det \begin{bmatrix} b+c & c+a & a+b \\ b_1+c_1 & c_1+a_1 & a_1+b_1 \\ b_2+c_2 & c_2+a_2 & a_2+b_2 \end{bmatrix} = 2 \det \begin{bmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}.$$

- 158 Evaluate the determinant of the following matrix by writing it as a sum of determinants

$$\begin{bmatrix} am+bp & an+bq \\ cm+dp & cn+dq \end{bmatrix}$$

- 159 In the case of each matrix, calculate the sum of all the algebraic co-factors of the individual elements in the expansion of the corresponding determinant:

$$\text{a) } \begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & a_n \end{bmatrix}; \quad \text{b) } \begin{bmatrix} 0 & 0 & \dots & 0 & a_1 \\ 0 & 0 & \dots & a_2 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & 0 & \dots & 0 & 0 \end{bmatrix}.$$

- 160 Expand the determinant of the following matrix by co-factors of the elements of the third row and carry out the computations

$$\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & -1 & -1 & 1 \\ a & b & c & d \\ -1 & -1 & 1 & 0 \end{bmatrix}.$$

- 161 Expand the determinant of the following matrix by co-factors of the elements of the last column and carry out the computations

$$\begin{bmatrix} 2 & 1 & 1 & x \\ 1 & 2 & 1 & y \\ 1 & 1 & 2 & z \\ 1 & 1 & 1 & t \end{bmatrix}.$$

- 162 Expand the determinant of the following matrix by co-factors of the elements of the first column and carry out the calculations

$$\begin{bmatrix} a & 1 & 1 & 1 \\ b & 0 & 1 & 1 \\ c & 1 & 0 & 1 \\ d & 1 & 1 & 0 \end{bmatrix}.$$

## 5. EVALUATION OF DETERMINANTS

Calculate the values of determinants of the following matrices:

$$\text{*163} \begin{bmatrix} 13547 & 13647 \\ 28423 & 28523 \end{bmatrix}, \quad \text{164} \begin{bmatrix} 246 & 427 & 327 \\ 1014 & 543 & 443 \\ -342 & 721 & 621 \end{bmatrix}.$$

$$\text{165} \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}, \quad \text{166} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}, \quad \text{167} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

$$168 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{bmatrix}.$$

$$169 \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 1 & -4 & 3 \\ 3 & -4 & -1 & 2 \\ 4 & 3 & -2 & -1 \end{bmatrix}.$$

$$170 \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 4 & 1 & 1 \\ 1 & 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 1 & 6 \end{bmatrix}.$$

$$171 \begin{bmatrix} 5 & 6 & 0 & 0 & 0 \\ 1 & 5 & 6 & 0 & 0 \\ 0 & 1 & 5 & 6 & 0 \\ 0 & 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}.$$

$$172 \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & a & b \\ 1 & a & 0 & c \\ 1 & b & c & 0 \end{bmatrix}.$$

$$173 \begin{bmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{bmatrix}.$$

$$174 \begin{bmatrix} x & 0 & -1 & 1 & 0 \\ 1 & x & -1 & 1 & 0 \\ 1 & 0 & x-1 & 0 & 1 \\ 0 & 1 & -1 & x & 1 \\ 0 & 1 & -1 & 0 & x \end{bmatrix}.$$

$$175 \begin{bmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+z & 1 \\ 1 & 1 & 1 & 1-z \end{bmatrix}.$$

$$176 \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 2-x^2 & 2 & 3 \\ 2 & 3 & 1 & 5 \\ 2 & 3 & 1 & 9-x^2 \end{bmatrix}.$$

$$177 \begin{bmatrix} \cos(a-b) & \cos(b-c) & \cos(c-a) \\ \cos(a+b) & \cos(b+c) & \cos(c+a) \\ \sin(a+b) & \sin(b+c) & \sin(c+a) \end{bmatrix}.$$

$$178 \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}.$$

$$*179 \begin{bmatrix} 1 & 2 & 3 & \dots & n \\ -1 & 0 & 3 & \dots & n \\ -1 & -2 & 0 & \dots & n \\ \dots & \dots & \dots & \dots & \dots \\ -1 & -2 & -3 & \dots & 0 \end{bmatrix}.$$

**\*180**

$$\begin{bmatrix} 1 & a_1 & a_2 & \dots & a_n \\ 1 & a_1 + b_1 & a_2 & \dots & a_n \\ 1 & a_1 & a_2 + b_2 & \dots & a_n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_1 & a_2 & \dots & a_n + b_n \end{bmatrix}.$$

**\*181**

$$\begin{bmatrix} 1 & x_1 & x_2 & \dots & x_{n-1} & x_n \\ 1 & x & x_2 & \dots & x_{n-1} & x_n \\ 1 & x_1 & x & \dots & x_{n-1} & x_n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x_1 & x_2 & \dots & x & x_n \\ 1 & x_1 & x_2 & \dots & x_{n-1} & x \end{bmatrix}.$$

**\*182**

$$\begin{bmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 1 & 3 & 3 & \dots & n-1 & n \\ 1 & 2 & 5 & \dots & n-1 & n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 2 & 3 & \dots & 2n-3 & n \\ 1 & 2 & 3 & \dots & n-1 & 2n-1 \end{bmatrix}.$$

**\*183**

$$\begin{bmatrix} 1 & 2 & 2 & \dots & 2 \\ 2 & 2 & 2 & \dots & 2 \\ 2 & 2 & 3 & \dots & 2 \\ \dots & \dots & \dots & \dots & \dots \\ 2 & 2 & 2 & \dots & n \end{bmatrix}.$$

**\*184**

$$\begin{bmatrix} 1 & b_1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 1-b_1 & b_2 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1-b_2 & b_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1-b_{n-1} & b_n \\ 0 & 0 & 0 & 0 & \dots & -1 & 1-b_n \end{bmatrix}.$$

**\*185**

$$\begin{bmatrix} a & a+h & a+2h & \dots & a+(n-1)h \\ -a & a & 0 & \dots & 0 \\ 0 & -a & a & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a \end{bmatrix}.$$

**\*186**

$$\begin{bmatrix} a-(a+h) & \dots & (-1)^{n-3}[a+(n-2)h] & (-1)^{n-1}[a+(n-1)h] \\ a & a & \dots & 0 \\ 0 & a & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a \end{bmatrix}.$$

\*187

$$\begin{bmatrix} 1 & C_1^n & C_2^n & C_3^n & \dots & C_{n-2}^n & C_{n-1}^n & C_n^n \\ 1 & C_1^{n-1} & C_2^{n-1} & C_3^{n-1} & \dots & C_{n-2}^{n-1} & C_{n-1}^{n-1} & 0 \\ 1 & C_1^{n-2} & C_2^{n-2} & C_3^{n-2} & \dots & C_{n-2}^{n-2} & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & C_1^2 & C_2^2 & 0 & \dots & 0 & 0 & 0 \\ 1 & C_1^1 & 0 & 0 & \dots & 0 & 0 & 0 \\ a_0 & a_1 & a_2 & a_3 & \dots & a_{n-2} & a_{n-1} & a_n \end{bmatrix}.$$

\*188

$$\begin{bmatrix} a_0 & -1 & 0 & \dots & 0 & 0 \\ a_1 & x & -1 & \dots & 0 & 0 \\ a_2 & 0 & x & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n-1} & 0 & 0 & \dots & x & -1 \\ a_n & 0 & 0 & \dots & 0 & x \end{bmatrix}.$$

\*189

$$\begin{bmatrix} n & n-1 & n-2 & \dots & 3 & 2 & 1 \\ -1 & x & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & x & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 & x & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 & x \end{bmatrix}.$$

\*190 Let  $f(x)$  be defined as the determinant of the following matrix

$$f(x) = \det \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & x \\ 1 & 2 & 0 & 0 & \dots & 0 & x^2 \\ 1 & 3 & 3 & 0 & \dots & 0 & x^3 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & n & C_2^n & C_3^n & \dots & C_{n-1}^n & x^n \\ 1 & n+1 & C_2^{n+1} & C_3^{n+1} & \dots & C_{n-1}^{n+1} & x^{n+1} \end{bmatrix}.$$

Calculate the value  $f(x+1) - f(x)$ .

Calculate the determinants of the following matrices:

$$*191 \begin{bmatrix} x & a_1 & a_2 & \dots & a_{n-1} & 1 \\ a_1 & x & a_2 & \dots & a_{n-1} & 1 \\ a_1 & a_2 & x & \dots & a_{n-1} & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_1 & a_2 & a_3 & \dots & x & 1 \\ a_1 & a_2 & a_3 & \dots & a_n & 1 \end{bmatrix}.$$

$$*192 \begin{bmatrix} x & a & a & \dots & a \\ a & x & a & \dots & a \\ a & a & x & \dots & a \\ \dots & \dots & \dots & \dots & \dots \\ a & a & a & \dots & x \end{bmatrix}.$$

$$193 \begin{bmatrix} x & a & a & \dots & a & a \\ -a & x & a & \dots & a & a \\ -a & -a & x & \dots & a & a \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -a & -a & -a & \dots & -a & x \end{bmatrix}.$$

$$*194 \begin{bmatrix} -a_1 & a_1 & 0 & \dots & 0 & 0 \\ 0 & -a_2 & a_2 & \dots & 0 & 0 \\ 0 & 0 & -a_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -a_n & a_n \\ 1 & 1 & 1 & \dots & 1 & 1 \end{bmatrix}.$$

$$*195 \begin{bmatrix} a_1 & -a_2 & 0 & \dots & 0 & 0 \\ 0 & a_2 & -a_3 & \dots & 0 & 0 \\ 0 & 0 & a_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{n-1} & -a_n \\ 1 & 1 & 1 & \dots & 1 & 1+a_n \end{bmatrix}.$$

$$*196 \begin{bmatrix} h & -1 & 0 & 0 & \dots & 0 \\ hx & h & -1 & 0 & \dots & 0 \\ hx^2 & hx & h & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ hx^n & hx^{n-1} & hx^{n-2} & hx^{n-3} & \dots & h \end{bmatrix}.$$

$$*197 \begin{bmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & x & \dots & x & x \\ 1 & x & 0 & \dots & x & x \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x & x & \dots & 0 & x \\ 1 & x & x & \dots & x & 0 \end{bmatrix}.$$

$$*198 \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & a_1+a_2 & \dots & a_1+a_n \\ 1 & a_2+a_1 & 0 & \dots & a_2+a_n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_n+a_1 & a_n+a_2 & \dots & 0 \end{bmatrix}.$$

$$*199 \begin{bmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 1 & 1 & 1 & \dots & 1 & 1-n \\ 1 & 1 & 1 & \dots & 1-n & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1-n & 1 & \dots & 1 & 1 \end{bmatrix}.$$

\*200 (This matrix has order  $n + 1$ .)

$$\begin{bmatrix} 2 & 1 - \frac{1}{n} & 1 - \frac{1}{n} & \dots & 1 - \frac{1}{n} \\ 1 - \frac{1}{n} & 2 & 1 - \frac{1}{n} & \dots & 1 - \frac{1}{n} \\ \dots & \dots & \dots & \dots & \dots \\ 1 - \frac{1}{n} & 1 - \frac{1}{n} & 1 - \frac{1}{n} & \dots & 2 \end{bmatrix}$$

\*201

$$\begin{bmatrix} 1 & a & a^2 & a^3 & \dots & a^n \\ x_{11} & 1 & a & a^2 & \dots & a^{n-1} \\ x_{21} & x_{22} & 1 & a & \dots & a^{n-2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x_{n1} & x_{n2} & x_{n3} & x_{n4} & \dots & 1 \end{bmatrix}.$$

\*202

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 2 & 1 & 2 & 3 & \dots & n-1 \\ 3 & 2 & 1 & 2 & \dots & n-2 \\ 4 & 3 & 2 & 1 & \dots & n-3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ n & n-1 & n-2 & n-3 & \dots & 1 \end{bmatrix}.$$

\*203

$$\begin{bmatrix} a_0 & b_1 & 0 & 0 & \dots & 0 & 0 \\ a_1 & -b_0 & b_2 & 0 & \dots & 0 & 0 \\ a_2 & 0 & -b_1 & b_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n-1} & 0 & 0 & 0 & \dots & -b_{n-2} & b_n \\ a_n & 0 & 0 & 0 & \dots & 0 & -b_{n-1} \end{bmatrix}.$$

\*204

$$\begin{bmatrix} a & a^2 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2a+b & (a+b)^2 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2a+3b & (a+2b)^2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2a+(2n-1)b & (a+nb)^2 \\ 0 & 0 & 0 & 0 & \dots & 1 & 2a+(2n+1)b \end{bmatrix}.$$



**\*205**

$$\begin{bmatrix} x & y & 0 & \dots & 0 & 0 \\ 0 & x & y & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & x & y \\ y & 0 & 0 & \dots & 0 & x \end{bmatrix}.$$

**\*206**

$$\begin{bmatrix} 1+x_1y_1 & 1+x_1y_2 & \dots & 1+x_1y_n \\ 1+x_2y_1 & 1+x_2y_2 & \dots & 1+x_2y_n \\ \cdot & \cdot & \cdot & \cdot \\ 1+x_ny_1 & 1+x_ny_2 & \dots & 1+x_ny_n \end{bmatrix}.$$

**207**

$$\begin{bmatrix} a_1-b_1 & a_1-b_2 & \dots & a_1-b_n \\ a_2-b_1 & a_2-b_2 & \dots & a_2-b_n \\ \cdot & \cdot & \cdot & \cdot \\ a_n-b_1 & a_n-b_2 & \dots & a_n-b_n \end{bmatrix}.$$

**\*208**

$$\begin{bmatrix} 1+a_1+x_1 & a_1+x_2 & \dots & a_1+x_n \\ a_2+x_1 & 1+a_2+x_2 & \dots & a_2+x_n \\ \cdot & \cdot & \cdot & \cdot \\ a_n+x_1 & a_n+x_2 & \dots & 1+a_n+x_n \end{bmatrix}.$$

**209**

$$\begin{bmatrix} a^n-\alpha & a^{n+1}-\alpha & \dots & a^{n+p-1}-\alpha \\ a^{n+p}-\alpha & a^{n+p+1}-\alpha & \dots & a^{n+2p-1}-\alpha \\ \cdot & \cdot & \cdot & \cdot \\ a^{n+p(p-1)}-\alpha & a^{n+p(p-1)+1}-\alpha & \dots & a^{n+p^2-1}-\alpha \end{bmatrix}.$$

- 210 Let  $f_1(x), f_2(x), \dots, f_n(x)$  be polynomials in  $x$  of degree not exceeding  $n-2$ ; let the numbers  $a_1, a_2, \dots, a_n$  be arbitrary. Show that the determinant of the following matrix is 0.

$$\begin{bmatrix} f_1(a_1) & f_1(a_2) & \dots & f_1(a_n) \\ f_2(a_1) & f_2(a_2) & \dots & f_2(a_n) \\ \cdot & \cdot & \cdot & \cdot \\ f_n(a_1) & f_n(a_2) & \dots & f_n(a_n) \end{bmatrix}$$

Calculate the determinants of the following matrices:

**\*211**

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \dots & n-1 & n \\ -1 & x & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & x & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & x \end{bmatrix}.$$

**\*212**

$$\begin{bmatrix} a_1 + x_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ -x_1 & x_2 & 0 & \dots & 0 & 0 \\ 0 & -x_2 & x_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -x_{n-1} & x_n \end{bmatrix}.$$

**\*213**

$$\begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n \\ -y_1 & x_1 & 0 & \dots & 0 & 0 \\ 0 & -y_2 & x_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -y_n & x_n \end{bmatrix}.$$

**\*214**

$$\begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & a_1 & 0 & \dots & 0 \\ 1 & 0 & a_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & a_n \end{bmatrix}.$$

**\*215**

$$\begin{bmatrix} n! a_0 & (n-1)! a_1 & (n-2)! a_2 & \dots & a_n \\ -n & x & 0 & \dots & 0 \\ 0 & -(n-1) & x & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & x \end{bmatrix}.$$

216

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & a_1 & 0 & 0 & 0 \\ 1 & 1 & a_2 & 0 & 0 \\ 1 & 0 & 1 & a_3 & 0 \\ 1 & 0 & 0 & 1 & a_4 \end{bmatrix}.$$

Generalize problem 216 to the case of an arbitrary  $n$ -th order matrix with the same pattern; namely 1 is in the first column, 1 in the upper right-hand corner, and 1 in the first diagonal.

Calculate the determinants of the following matrices:

217

$$\begin{bmatrix} \alpha + \beta & \alpha\beta & 0 & \dots & 0 & 0 \\ 1 & \alpha + \beta & \alpha\beta & \dots & 0 & 0 \\ 0 & 1 & \alpha + \beta & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & \alpha + \beta \end{bmatrix}.$$

218

$$\begin{bmatrix} 2 & 1 & 0 & 0 & \dots & 0 \\ 1 & 2 & 1 & 0 & \dots & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2 \end{bmatrix}.$$

\*219

$$\begin{bmatrix} 2 \cos \theta & 1 & 0 & \dots & 0 & 0 \\ 1 & 2 \cos \theta & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 2 \cos \theta \end{bmatrix}.$$

220

$$\begin{bmatrix} \cos \theta & 1 & 0 & 0 \\ 1 & 2 \cos \theta & 1 & 0 \\ 0 & 1 & 2 \cos \theta & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 2 \cos \theta \end{bmatrix}.$$

\*221

$$\begin{bmatrix} x & 1 & 0 & \dots & 0 \\ 1 & x & 1 & \dots & 0 \\ 0 & 1 & x & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & x \end{bmatrix}.$$

\*222

$$\begin{bmatrix} x_1 y_1 & x_1 y_2 & x_1 y_3 & \dots & x_1 y_n \\ x_1 y_2 & x_2 y_2 & x_2 y_3 & \dots & x_2 y_n \\ x_1 y_3 & x_2 y_3 & x_3 y_3 & \dots & x_3 y_n \\ \dots & \dots & \dots & \dots & \dots \\ x_1 y_n & x_2 y_n & x_3 y_n & \dots & x_n y_n \end{bmatrix}.$$

\*223

$$\begin{bmatrix} 1 + a_1 & 1 & 1 & \dots & 1 \\ 1 & 1 + a_2 & 1 & \dots & 1 \\ 1 & 1 & 1 + a_3 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 1 + a_n \end{bmatrix}.$$

224

$$\begin{bmatrix} 1 & 1 & \dots & 1 & a_1 + 1 \\ 1 & 1 & \dots & a_2 + 1 & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_{n-1} + 1 & \dots & 1 & 1 \\ a_n + 1 & 1 & \dots & 1 & 1 \end{bmatrix}.$$

\*225

$$\begin{bmatrix} a_1 & x & x & \dots & x \\ x & a_2 & x & \dots & x \\ x & x & a_3 & \dots & x \\ \dots & \dots & \dots & \dots & \dots \\ x & x & x & \dots & a_n \end{bmatrix}.$$

$$\begin{aligned}
 & *226 \begin{bmatrix} x_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ a_1 & x_2 & a_3 & \dots & a_{n-1} & a_n \\ a_1 & a_2 & x_3 & \dots & a_{n-1} & a_n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_1 & a_2 & a_3 & \dots & x_{n-1} & a_n \\ a_1 & a_2 & a_3 & \dots & a_{n-1} & x_n \end{bmatrix} & *227 \begin{bmatrix} x_1 & a_2 b_1 & a_3 b_1 & \dots & a_n b_1 \\ a_1 b_2 & x_2 & a_3 b_2 & \dots & a_n b_2 \\ a_1 b_3 & a_2 b_3 & x_3 & \dots & a_n b_3 \\ \dots & \dots & \dots & \dots & \dots \\ a_1 b_n & a_2 b_n & a_3 b_n & \dots & x_n \end{bmatrix}
 \end{aligned}$$

$$*228 \begin{bmatrix} x_1 - m & x_2 & x_3 & \dots & x_n \\ x_1 & x_2 - m & x_3 & \dots & x_n \\ x_1 & x_2 & x_3 - m & \dots & x_n \\ \dots & \dots & \dots & \dots & \dots \\ x_1 & x_2 & x_3 & \dots & x_n - m \end{bmatrix}$$

229 Solve the following equation for  $x$ :

$$\det \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} & a_n - \alpha_n x \\ a_1 & a_2 & \dots & a_{n-1} - \alpha_{n-1} x & a_n \\ \dots & \dots & \dots & \dots & \dots \\ a_1 - \alpha_1 x & a_2 & \dots & a_{n-1} & a_n \end{bmatrix} = 0.$$

Calculate the determinants of the following matrices:

$$\begin{aligned}
 & *230 \begin{bmatrix} a & 0 & 0 & \dots & 0 & 0 & b \\ 0 & a & 0 & \dots & 0 & b & 0 \\ 0 & 0 & a & \dots & b & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & b & \dots & a & 0 & 0 \\ 0 & b & 0 & \dots & 0 & a & 0 \\ b & 0 & 0 & \dots & 0 & 0 & a \end{bmatrix} & *231 \begin{bmatrix} 1 & -b & -b & -b & \dots & -b \\ 1 & na & -2b & -3b & \dots & -(n-1)b \\ 1 & (n-1)a & a & -3b & \dots & -(n-1)b \\ 1 & (n-2)a & a & a & \dots & -(n-1)b \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 2a & a & a & \dots & a \end{bmatrix}
 \end{aligned}$$

(Order  $2n$ )

**\*232**

$$\begin{bmatrix} (x-a_1)^2 & a_2^2 & \dots & a_n^2 \\ a_1^2 & (x-a_2)^2 & \dots & a_n^2 \\ \dots & \dots & \dots & \dots \\ a_1^2 & a_2^2 & \dots & (x-a_n)^2 \end{bmatrix}.$$

**\*233**

$$\begin{bmatrix} (x-a_1)^2 & a_1a_2 & \dots & a_1a_n \\ a_1a_2 & (x-a_2)^2 & \dots & a_2a_n \\ \dots & \dots & \dots & \dots \\ a_1a_n & a_2a_n & \dots & (x-a_n)^2 \end{bmatrix}.$$

**\*234**

$$\begin{bmatrix} 1-b_1 & b_2 & 0 & 0 & \dots & 0 \\ -1 & 1-b_2 & b_3 & 0 & \dots & 0 \\ 0 & -1 & 1-b_3 & b_4 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1-b_n \end{bmatrix}.$$

**\*235**

$$\begin{bmatrix} 0 & a_2 & a_3 & a_4 & \dots & a_{n-1} & a_n \\ b_1 & 0 & a_3 & a_4 & \dots & a_{n-1} & a_n \\ b_1 & b_2 & 0 & a_4 & \dots & a_{n-1} & a_n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_1 & b_2 & b_3 & b_4 & \dots & 0 & a_n \\ b_1 & b_2 & b_3 & b_4 & \dots & b_{n-1} & 0 \end{bmatrix}.$$

**\*236**

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 1 & 2 & 3 & 4 & \dots & n-1 \\ 1 & x & 1 & 2 & 3 & \dots & n-2 \\ 1 & x & x & 1 & 2 & \dots & n-3 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x & x & x & x & \dots & 1 \end{bmatrix}.$$

**\*237**

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \dots & n \\ x & 1 & 2 & 3 & \dots & n-1 \\ x & x & 1 & 2 & \dots & n-2 \\ x & x & x & 1 & \dots & n-3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x & x & x & x & \dots & 1 \end{bmatrix}.$$

**\*238**

$$\begin{bmatrix} a_0x^n & a_1x^{n-1} & a_2x^{n-2} & \dots & a_{n-1}x & a_n \\ a_0x & b_1 & 0 & \dots & 0 & 0 \\ a_0x^2 & a_1x & b_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_0x^{n-1} & a_1x^{n-2} & a_2x^{n-3} & \dots & b_{n-1} & 0 \\ a_0x^n & a_1x^{n-1} & a_2x^{n-2} & \dots & a_{n-1}x & b_n \end{bmatrix}.$$

\*239 Show

$$\det \begin{bmatrix} a_{00}x^n & a_{01}x^{n-1} & a_{02}x^{n-2} & \dots & a_{0n} \\ a_{10}x & a_{11} & 0 & \dots & 0 \\ a_{20}x^2 & a_{21}x & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n0}x^n & a_{n1}x^{n-1} & a_{n2}x^{n-2} & \dots & a_{nn} \end{bmatrix} = x^n \cdot \det \begin{bmatrix} a_{00} & a_{01} & a_{02} & \dots & a_{0n} \\ a_{10} & a_{11} & 0 & \dots & 0 \\ a_{20} & a_{21} & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n0} & a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}.$$

Calculate the determinants of the following matrices:

\*240

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & C_1^2 & C_1^3 & \dots & C_1^n \\ 1 & C_2^3 & C_2^4 & \dots & C_2^{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & C_{n-1}^n & C_{n-1}^{n+1} & \dots & C_{n-1}^{2n-2} \end{bmatrix}.$$

\*241

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ C_1^m & C_1^{m+1} & \dots & C_1^{m+n} \\ C_2^{m+1} & C_2^{m+2} & \dots & C_2^{m+n+1} \\ \dots & \dots & \dots & \dots \\ C_n^{m+n-1} & C_n^{m+n} & \dots & C_n^{m+2n-1} \end{bmatrix}.$$

\*242

$$\begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & C_1^2 C_2^2 & 0 & \dots & 0 \\ 1 & C_1^3 C_2^3 & C_3^3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & C_1^n C_2^n & C_3^n & \dots & C_{n-1}^n \end{bmatrix}.$$

\*243

$$\begin{bmatrix} C_k^m & C_{k+1}^m & \dots & C_{k+n}^m \\ C_k^{m+1} & C_{k+1}^{m+1} & \dots & C_{k+n}^{m+1} \\ \dots & \dots & \dots & \dots \\ C_k^{m+n} & C_{k+1}^{m+n} & \dots & C_{k+n}^{m+n} \end{bmatrix}.$$

\*244

$$\begin{bmatrix} C_m^{k+m} & C_m^{k+m+1} & \dots & C_m^{k+2m} \\ C_m^{k+m+1} & C_m^{k+m+2} & \dots & C_m^{k+2m+1} \\ \dots & \dots & \dots & \dots \\ C_m^{k+2m} & C_m^{k+2m+1} & \dots & C_m^{k+3m} \end{bmatrix}.$$

\*245

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & C_1^1 & 0 & \dots & 0 & x \\ 1 & C_1^2 & C_2^2 & \dots & 0 & x^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & C_1^n & C_2^n & \dots & C_{n-1}^n & x^n \end{bmatrix}.$$

$$*246 \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 1 \\ 1 & 1! & 0 & 0 & \dots & x \\ 1 & 2 & 2! & 0 & \dots & x^2 \\ 1 & 3 & 3 \cdot 2 & 3! & \dots & x^3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & n & n(n-1) & n(n-1)(n-2) & \dots & x^n \end{bmatrix}.$$

$$*247 \begin{bmatrix} \alpha & \alpha + \delta & \alpha + 2\delta & \alpha + 3\delta & \dots & \alpha + (n-1)\delta \\ \alpha & 2\alpha + \delta & 3\alpha + 3\delta & 4\alpha + 6\delta & \dots & C_1^n \alpha + C_2^n \delta \\ \alpha & 3\alpha + \delta & 6\alpha + 4\delta & 10\alpha + 10\delta & \dots & C_2^{n+1} \alpha + C_3^{n+1} \delta \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha & C_{n-1}^n \alpha + \delta & C_{n-1}^{n+1} \alpha + C_n^{n+1} \delta & C_{n-1}^{n+2} \alpha + C_n^{n+2} \delta & \dots & C_{n-1}^{n-2} \alpha + C_n^{n-2} \delta \end{bmatrix}.$$

$$*248 \begin{bmatrix} x & y & y & \dots & y & y \\ z & x & y & \dots & y & y \\ z & z & x & \dots & y & y \\ \dots & \dots & \dots & \dots & \dots & \dots \\ z & z & z & \dots & x & y \\ z & z & z & \dots & z & x \end{bmatrix}.$$

$$*249 \begin{bmatrix} a & a & a & \dots & a & 0 \\ a & a & a & \dots & 0 & b \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a & 0 & b & \dots & b & b \\ 0 & b & b & \dots & b & b \end{bmatrix}.$$

$$250 \begin{bmatrix} a_1 & x & x & \dots & x \\ y & a_2 & x & \dots & x \\ \dots & \dots & \dots & \dots & \dots \\ y & y & y & \dots & a_n \end{bmatrix}.$$

$$251 \begin{bmatrix} c_1 & a & a & \dots & a & 1 \\ b & c_2 & a & \dots & a & 1 \\ b & b & c_3 & \dots & a & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b & b & b & \dots & c_n & 1 \\ 1 & 1 & 1 & \dots & 1 & 0 \end{bmatrix}.$$

$$*252 \begin{bmatrix} \lambda & a & a & a & \dots & a \\ b & a & \beta & \beta & \dots & \beta \\ b & \beta & a & \beta & \dots & \beta \\ b & \beta & \beta & a & \dots & \beta \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b & \beta & \beta & \beta & \dots & a \end{bmatrix}.$$

$$*253 \begin{bmatrix} 1 & 2 & 3 & \dots & n \\ 2 & 3 & 4 & \dots & 1 \\ 3 & 4 & 5 & \dots & 2 \\ \dots & \dots & \dots & \dots & \dots \\ n & 1 & 2 & \dots & n-1 \end{bmatrix}.$$

$$*254 \begin{bmatrix} a & a+h & a+2h & \dots & a+(n-1)h \\ a+h & a+2h & a+3h & \dots & a \\ a+2h & a+3h & a+4h & \dots & a+h \\ \dots & \dots & \dots & \dots & \dots \\ a+(n-1)h & a & a+h & \dots & a+(n-2)h \end{bmatrix}.$$

$$255 \quad \begin{bmatrix} 1 & x & x^2 & \dots & x^{n-1} \\ x^{n-1} & 1 & x & \dots & x^{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ x & x^2 & x^3 & \dots & 1 \end{bmatrix}.$$

$$*256 \quad \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix}.$$

$$257 \quad \begin{bmatrix} a & b & c & d & e & f & g & h \\ b & a & d & c & f & e & h & g \\ c & d & a & b & g & h & e & f \\ d & c & b & a & h & g & f & e \\ e & f & g & h & a & b & c & d \\ f & e & h & g & b & a & d & c \\ g & h & e & f & c & d & a & b \\ h & g & f & e & d & c & b & a \end{bmatrix}.$$

$$*258 \quad \begin{bmatrix} x & a_1 & a_2 & \dots & a_n \\ a_1 & x & a_2 & \dots & a_n \\ \dots & \dots & \dots & \dots & \dots \\ a_1 & a_2 & a_3 & \dots & x \end{bmatrix}.$$

(order  $n + 1$ )

$$*259 \quad \begin{bmatrix} \cos^{n-1} \varphi_1 & \cos^{n-2} \varphi_1 & \dots & \cos \varphi_1 & 1 \\ \cos^{n-1} \varphi_2 & \cos^{n-2} \varphi_2 & \dots & \cos \varphi_2 & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \cos^{n-1} \varphi_n & \cos^{n-2} \varphi_n & \dots & \cos \varphi_n & 1 \end{bmatrix}.$$

$$260 \quad \begin{bmatrix} 1 & 1 & \dots & 1 \\ \sin \varphi_1 & \sin \varphi_2 & \dots & \sin \varphi_n \\ \sin^2 \varphi_1 & \sin^2 \varphi_2 & \dots & \sin^2 \varphi_n \\ \dots & \dots & \dots & \dots \\ \sin^{n-1} \varphi_1 & \sin^{n-1} \varphi_2 & \dots & \sin^{n-1} \varphi_n \end{bmatrix}.$$

$$261 \quad \begin{bmatrix} a^n (a-1)^n & \dots & (a-n)^n \\ a^{n-1} (a-1)^{n-1} & \dots & (a-n)^{n-1} \\ \dots & \dots & \dots \\ a & a-1 & \dots & a-n \\ 1 & 1 & \dots & 1 \end{bmatrix}.$$

$$262 \quad \begin{bmatrix} (a_1+x)^n & (a_1+x)^{n-1} & \dots & a_1+x & 1 \\ (a_2+x)^n & (a_2+x)^{n-1} & \dots & a_2+x & 1 \\ \dots & \dots & \dots & \dots & \dots \\ (a_{n+1}+x)^n & (a_{n+1}+x)^{n-1} & \dots & a_{n+1}+x & 1 \end{bmatrix}.$$

$$263 \quad \begin{bmatrix} (2n-1)^n & (2n-2)^n & \dots & n^n & (2n)^n \\ (2n-1)^{n-1} & (2n-2)^{n-1} & \dots & n^{n-1} & (2n)^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 2n-1 & 2n-2 & \dots & n & 2n \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix}.$$

$$*264 \quad \begin{bmatrix} w_1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ w_2 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ w_n & a_n & a_n^2 & \dots & a_n^{n-1} \end{bmatrix}.$$



\*265

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ x_1 + 1 & x_2 + 1 & x_3 + 1 & \dots & x_n + 1 \\ x_1^2 + x_1 & x_2^2 + x_2 & x_3^2 + x_3 & \dots & x_n^2 + x_n \\ \dots & \dots & \dots & \dots & \dots \\ x_1^{n-1} + x_1^{n-2} & x_2^{n-1} + x_2^{n-2} & x_3^{n-1} + x_3^{n-2} & \dots & x_n^{n-1} + x_n^{n-2} \end{bmatrix}.$$

266

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 + \sin \varphi_1 & 1 + \sin \varphi_2 & \dots & 1 + \sin \varphi_n \\ \sin \varphi_1 + \sin^2 \varphi_1 & \sin \varphi_2 + \sin^2 \varphi_2 & \dots & \sin \varphi_n + \sin^2 \varphi_n \\ \dots & \dots & \dots & \dots \\ \sin^{n-2} \varphi_1 + \sin^{n-1} \varphi_1 & \sin^{n-2} \varphi_2 + \sin^{n-1} \varphi_2 & \dots & \sin^{n-2} \varphi_n + \sin^{n-1} \varphi_n \end{bmatrix}.$$

267

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ \varphi_1(x_1) & \varphi_1(x_2) & \dots & \varphi_1(x_n) \\ \varphi_2(x_1) & \varphi_2(x_2) & \dots & \varphi_2(x_n) \\ \dots & \dots & \dots & \dots \\ \varphi_{n-1}(x_1) & \varphi_{n-1}(x_2) & \dots & \varphi_{n-1}(x_n) \end{bmatrix},$$

where  $\varphi_k(x) = x^k + a_{1k}x^{k-1} + \dots + a_{kk}$ .

268

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ F_1(\cos \varphi_1) & F_1(\cos \varphi_2) & \dots & F_1(\cos \varphi_n) \\ F_2(\cos \varphi_1) & F_2(\cos \varphi_2) & \dots & F_2(\cos \varphi_n) \\ \dots & \dots & \dots & \dots \\ F_{n-1}(\cos \varphi_1) & F_{n-1}(\cos \varphi_2) & \dots & F_{n-1}(\cos \varphi_n) \end{bmatrix},$$

where  $F_k(x) = a_{0k}x^k + a_{1k}x^{k-1} + \dots + a_{kk}$ .

\*269

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ \binom{x_1}{1} & \binom{x_2}{1} & \dots & \binom{x_n}{1} \\ \binom{x_1}{2} & \binom{x_2}{2} & \dots & \binom{x_n}{2} \\ \dots & \dots & \dots & \dots \\ \binom{x_1}{n-1} & \binom{x_2}{n-1} & \dots & \binom{x_n}{n-1} \end{bmatrix},$$

$$\text{where } \binom{x}{k} = \frac{x(x-1)\dots(x-k+1)}{1 \cdot 2 \dots k}.$$

\*270 If  $a_1, a_2, \dots, a_n$  are rational integers, show that the quantity  $1^{n-1} 2^{n-2} \dots (n-1)$  is a factor of the determinant of the matrix

$$\begin{bmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-1} \end{bmatrix}$$

Calculate the determinants of the following matrices:

$$\begin{array}{ll} \text{*271} \begin{bmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 2^3 & 3^3 & \dots & n^3 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 2^{2n-1} & 3^{2n-1} & \dots & n^{2n-1} \end{bmatrix} & \text{*272} \begin{bmatrix} \frac{x_1}{x_1-1} & \frac{x_2}{x_2-1} & \dots & \frac{x_n}{x_n-1} \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \dots & \dots & \dots & \dots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{bmatrix} \end{array}$$

$$\text{*273} \begin{bmatrix} a_1^n & a_1^{n-1}b_1 & a_1^{n-2}b_1^2 & \dots & a_1b_1^{n-1} & b_1^n \\ a_2^n & a_2^{n-1}b_2 & a_2^{n-2}b_2^2 & \dots & a_2b_2^{n-1} & b_2^n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n+1}^n & a_{n+1}^{n-1}b_{n+1} & a_{n+1}^{n-2}b_{n+1}^2 & \dots & a_{n+1}b_{n+1}^{n-1} & b_{n+1}^n \end{bmatrix}.$$

274

$$\begin{bmatrix} \sin^{n-1} \alpha_1 & \sin^{n-2} \alpha_1 \cos \alpha_1 & \dots & \sin \alpha_1 \cos^{n-2} \alpha_1 & \cos^{n-1} \alpha_1 \\ \sin^{n-1} \alpha_2 & \sin^{n-2} \alpha_2 \cos \alpha_2 & \dots & \sin \alpha_2 \cos^{n-2} \alpha_2 & \cos^{n-1} \alpha_2 \\ \dots & \dots & \dots & \dots & \dots \\ \sin^{n-1} \alpha_n & \sin^{n-2} \alpha_n \cos \alpha_n & \dots & \sin \alpha_n \cos^{n-2} \alpha_n & \cos^{n-1} \alpha_n \end{bmatrix}.$$

\*275

$$\begin{bmatrix} a_1^{2n} + 1 & a_1^{2n-1} + a_1 & a_1^{2n-2} + a_1^2 & \dots & a_1^{n+1} + a_1^{n-1} & a_1^n \\ a_2^{2n} + 1 & a_2^{2n-1} + a_2 & a_2^{2n-2} + a_2^2 & \dots & a_2^{n+1} + a_2^{n-1} & a_2^n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n+1}^{2n} + 1 & a_{n+1}^{2n-1} + a_{n+1} & a_{n+1}^{2n-2} + a_{n+1}^2 & \dots & a_{n+1}^{n+1} + a_{n+1}^{n-1} & a_{n+1}^n \end{bmatrix}.$$

\*276

$$\begin{bmatrix} 1 & \cos \varphi_0 & \cos 2\varphi_0 & \dots & \cos(n-1)\varphi_0 \\ 1 & \cos \varphi_1 & \cos 2\varphi_1 & \dots & \cos(n-1)\varphi_1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \cos \varphi_{n-1} & \cos 2\varphi_{n-1} & \dots & \cos(n-1)\varphi_{n-1} \end{bmatrix}.$$

\*277

$$\begin{bmatrix} \sin(n+1)\alpha_0 & \sin n\alpha_0 & \dots & \sin \alpha_0 \\ \sin(n+1)\alpha_1 & \sin n\alpha_1 & \dots & \sin \alpha_1 \\ \dots & \dots & \dots & \dots \\ \sin(n+1)\alpha_n & \sin n\alpha_n & \dots & \sin \alpha_n \end{bmatrix}.$$

\*278

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1(x_1-1) & x_2(x_2-1) & \dots & x_n(x_n-1) \\ x_1^2(x_1-1) & x_2^2(x_2-1) & \dots & x_n^2(x_n-1) \\ \dots & \dots & \dots & \dots \\ x_1^{n-1}(x_1-1) & x_2^{n-1}(x_2-1) & \dots & x_n^{n-1}(x_n-1) \end{bmatrix}.$$

\*279

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ x_1^3 & x_2^3 & \dots & x_n^3 \\ \dots & \dots & \dots & \dots \\ x_1^n & x_2^n & \dots & x_n^n \end{bmatrix}.$$

\*280

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \dots & \dots & \dots & \dots \\ x_1^{n-2} & x_2^{n-2} & \dots & x_n^{n-2} \\ x_1^n & x_2^n & \dots & x_n^n \end{bmatrix}.$$

$$281 \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \dots & \dots & \dots & \dots \\ x_1^{s-1} & x_2^{s-1} & \dots & x_n^{s-1} \\ x_1^{s+1} & x_2^{s+1} & \dots & x_n^{s+1} \\ x_1^n & x_2^n & \dots & x_n^n \end{bmatrix} \quad *282 \begin{bmatrix} 1+x_1 & 1+x_1^2 & \dots & 1+x_1^n \\ 1+x_2 & 1+x_2^2 & \dots & 1+x_2^n \\ \dots & \dots & \dots & \dots \\ 1+x_n & 1+x_n^2 & \dots & 1+x_n^n \end{bmatrix}.$$

$$283 \begin{bmatrix} 1 & x & x^2 & x^3 \\ x^3 & x^2 & x & 1 \\ 1 & 2x & 3x^2 & 4x^3 \\ 4x^3 & 3x^2 & 2x & 1 \end{bmatrix} \quad 284 \begin{bmatrix} 1 & x & x^2 & x^3 & x^4 \\ 1 & 2x & 3x^2 & 4x^3 & 5x^4 \\ 1 & 4x & 9x^2 & 16x^3 & 25x^4 \\ 1 & y & y^2 & y^3 & y^4 \\ 1 & 2y & 3y^2 & 4y^3 & 5y^4 \end{bmatrix}.$$

$$*285 \begin{bmatrix} 1 & x & x^2 & \dots & x^n \\ 1 & 2x & 3x^2 & \dots & (n+1)x^n \\ 1 & 2^2x & 3^2x^2 & \dots & (n+1)^2x^n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 2^{n-1}x & 3^{n-1}x^2 & \dots & (n+1)^{n-1}x^n \\ 1 & y & y^2 & \dots & y^n \end{bmatrix}.$$

$$*286 \begin{bmatrix} 1 & x & x^2 & \dots & x^{n-1} \\ 1 & 2x & 3x^2 & \dots & nx^{n-1} \\ 1 & 2^2x & 3^2x^2 & \dots & n^2x^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 2^{k-1}x & 3^{k-1}x^2 & \dots & n^{k-1}x^{n-1} \\ 1 & y_1 & y_1^2 & \dots & y_1^{n-1} \\ 1 & y_2 & y_2^2 & \dots & y_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & y_{n-k} & y_{n-k}^2 & \dots & y_{n-k}^{n-1} \end{bmatrix}.$$

$$*287 \begin{bmatrix} 1 & x & x^2 & \dots & x^{n-1} \\ 0 & 1 & C_1^2x & \dots & C_1^{n-1}x^{n-2} \\ 0 & 0 & 1 & \dots & C_2^{n-1}x^{n-3} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & C_{k-1}^{n-1}x^{n-k} \\ 1 & y & y^2 & \dots & y^{n-1} \\ 0 & 1 & C_1^2y & \dots & C_1^{n-1}y^{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & C_{n-k-1}^{n-1}y^k \end{bmatrix}.$$

288 a) Write down the Laplace expansion of the determinant of a 4-th order matrix by minors of the first two rows.

b) Expand the determinant of the matrix

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix},$$

by some set of minors of order 2.

c) The same for the determinant of the matrix

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix},$$

d) Use the same method for the matrix of problem 145.

Calculate the determinants of the following matrices:

$$\text{e) } \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 4 & 0 & 0 & 0 \\ 3 & 6 & 10 & 0 & 0 & 0 \\ 4 & 9 & 14 & 1 & 1 & 1 \\ 5 & 15 & 24 & 1 & 5 & 9 \\ 9 & 24 & 38 & 1 & 25 & 81 \end{bmatrix}; \quad \text{f) } \begin{bmatrix} a_1 & 0 & b_1 & 0 \\ 0 & c_1 & 0 & d_1 \\ b_2 & 0 & a_2 & 0 \\ 0 & d_2 & 0 & c_2 \end{bmatrix};$$

$$g) \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ x_1 & x_2 & 0 & 0 & 0 & x_3 \\ a_1 & b_1 & 1 & 1 & 1 & c_1 \\ a_2 & b_2 & x_1 & x_2 & x_3 & c_2 \\ a_3 & b_3 & x_1^2 & x_2^2 & x_3^2 & c_3 \\ x_1^2 & x_2^2 & 0 & 0 & 0 & x_3^2 \end{bmatrix}; \quad h) \begin{bmatrix} \lambda & 0 & 0 & \dots & 0 & a \\ x_1 & \alpha & \beta & \dots & \beta & y_1 \\ x_2 & \beta & \alpha & \dots & \beta & y_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x_n & \beta & \beta & \dots & \alpha & y_n \\ a & 0 & 0 & \dots & 0 & \lambda \end{bmatrix}.$$

i) Expand the determinant of the matrix of problem 230 by the method of Laplace.

j) The same for the matrix of problem 171.

k) Calculate the determinant of the following matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & x_1 & x_2 & x_3 & x_4 \\ 0 & x_1^2 & x_2^2 & x_3^2 & x_4^2 \end{bmatrix}.$$

l) Let A, B, C, D, be the determinants of the respective third order matrices obtained by omitting the first, second, third, and fourth columns in the following array.

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix}$$

Establish the following identity:

$$\det \begin{bmatrix} a_1 & b_1 & c_1 & d_1 & 0 & 0 \\ a_2 & b_2 & c_2 & d_2 & 0 & 0 \\ a_3 & b_3 & c_3 & d_3 & 0 & 0 \\ 0 & 0 & a_1 & b_1 & c_1 & d_1 \\ 0 & 0 & a_2 & b_2 & c_2 & d_2 \\ 0 & 0 & a_3 & b_3 & c_3 & d_3 \end{bmatrix} = AD - BC.$$

\*m) A certain fifteenth order matrix has the form

$$\begin{bmatrix} \Delta & \Delta_1 & \Delta_1 \\ \Delta_1 & \Delta & \Delta_1 \\ \Delta_1 & \Delta_1 & \Delta \end{bmatrix},$$

where  $\Delta$ ,  $\Delta_1$  are given by

$$\Delta = \begin{pmatrix} a & x & x & -x & -x \\ x & 2a & a & 0 & 0 \\ x & a & 2a & 0 & 0 \\ -x & 0 & 0 & 2a & a \\ -x & 0 & 0 & a & 2a \end{pmatrix}, \quad \Delta_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

Calculate the determinant of the original matrix.

## 6. MULTIPLICATION OF DETERMINANTS

289 The following expressions are intended to represent products of determinants. Calculate the value of each expression by multiplying together a pair of matrices.

$$a) \quad \begin{vmatrix} 4 & 3 \\ 1 & 3 \end{vmatrix} \cdot \begin{vmatrix} 1 & -2 \\ -3 & 2 \end{vmatrix};$$

$$b) \quad \begin{vmatrix} 3 & 2 & 5 \\ -1 & 3 & 6 \\ 1 & -1 & 2 \end{vmatrix} \cdot \begin{vmatrix} -2 & 3 & 4 \\ -1 & -3 & 5 \\ 2 & 1 & -1 \end{vmatrix};$$

$$c) \quad \begin{vmatrix} 2 & 1 & 1 & 1 \\ -1 & 2 & 1 & 1 \\ -1 & -1 & 2 & 1 \\ -1 & -1 & -1 & 2 \end{vmatrix} \cdot \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} \cdot \begin{vmatrix} -2 & 1 \\ -1 & 2 \end{vmatrix}.$$

290 First multiply the matrices  $\Delta$ ,  $\delta$ , and in that way compute the product  $\det \Delta \cdot \det \delta$ .

$$a) \quad \Delta = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & -3 & -8 \\ -1 & 1 & 0 & -13 \\ 2 & 3 & 5 & 15 \end{bmatrix}, \quad \delta = \begin{bmatrix} 1 & -2 & -3 & -11 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$$b) \quad \Delta = \begin{bmatrix} -1 & -9 & -2 & 3 \\ -5 & 5 & 3 & -2 \\ -12 & -6 & 1 & 1 \\ 9 & 0 & -2 & 1 \end{bmatrix}, \quad \delta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix};$$

$$c) \quad \Delta = \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix}, \quad \delta = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$



- 291 Square each of the following matrices and that way compute the value of the determinant of the original matrix:

$$\begin{aligned} \text{a) } & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}; & \text{b) } & \begin{bmatrix} 1 & -1 & 1 & -1 \\ 2 & 2 & 1 & 1 \\ 2 & 0 & -3 & -1 \\ 3 & -7 & -1 & 9 \end{bmatrix}; \\ \text{c) } & \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix}. \end{aligned}$$

- 292 Define  $D$  by

$$\det \begin{bmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0n-1} \\ a_{10} & a_{11} & a_{12} & \cdots & a_{1n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n-1,0} & a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{bmatrix} = D.$$

Let  $\varphi_i(x) = a_{0i} + a_{1i}x + \cdots + a_{n-1,i}x^{n-1}$

Find the value of

$$\det \begin{bmatrix} \varphi_0(x_1) & \varphi_0(x_2) & \cdots & \varphi_0(x_n) \\ \varphi_1(x_1) & \varphi_1(x_2) & \cdots & \varphi_1(x_n) \\ \cdot & \cdot & \cdot & \cdot \\ \varphi_{n-1}(x_1) & \varphi_{n-1}(x_2) & \cdots & \varphi_{n-1}(x_n) \end{bmatrix}.$$

Show how this can be used to solve problems 265, 267, 268.

Calculate the determinants of the following matrices:

**\*293**

a) 
$$\begin{bmatrix} (b_0 + a_0)^n & (b_1 + a_0)^n & \dots & (b_n + a_0)^n \\ (b_0 + a_1)^n & (b_1 + a_1)^n & \dots & (b_n + a_1)^n \\ \dots & \dots & \dots & \dots \\ (b_0 + a_n)^n & (b_1 + a_n)^n & \dots & (b_n + a_n)^n \end{bmatrix};$$

b) 
$$\begin{bmatrix} \frac{1 - \alpha_1^n \beta_1^n}{1 - \alpha_1 \beta_1} & \frac{1 - \alpha_1^n \beta_2^n}{1 - \alpha_1 \beta_2} & \dots & \frac{1 - \alpha_1^n \beta_n^n}{1 - \alpha_1 \beta_n} \\ \frac{1 - \alpha_2^n \beta_1^n}{1 - \alpha_2 \beta_1} & \frac{1 - \alpha_2^n \beta_2^n}{1 - \alpha_2 \beta_2} & \dots & \frac{1 - \alpha_2^n \beta_n^n}{1 - \alpha_2 \beta_n} \\ \dots & \dots & \dots & \dots \\ \frac{1 - \alpha_n^n \beta_1^n}{1 - \alpha_n \beta_1} & \frac{1 - \alpha_n^n \beta_2^n}{1 - \alpha_n \beta_2} & \dots & \frac{1 - \alpha_n^n \beta_n^n}{1 - \alpha_n \beta_n} \end{bmatrix}.$$

**\*294**

$$\begin{bmatrix} \sin 2\alpha_1 & \sin (\alpha_1 + \alpha_2) & \dots & \sin (\alpha_1 + \alpha_n) \\ \sin (\alpha_2 + \alpha_1) & \sin 2\alpha_2 & \dots & \sin (\alpha_2 + \alpha_n) \\ \dots & \dots & \dots & \dots \\ \sin (\alpha_n + \alpha_1) & \sin (\alpha_n + \alpha_2) & \dots & \sin 2\alpha_n \end{bmatrix}.$$

**\*295**

$$\begin{bmatrix} s_0 & s_1 & s_2 & \dots & s_{n-1} & 1 \\ s_1 & s_2 & s_3 & \dots & s_n & x \\ \dots & \dots & \dots & \dots & \dots & \dots \\ s_{n-1} & s_n & s_{n+1} & \dots & s_{2n-2} & x^{n-1} \\ s_n & s_{n+1} & s_{n+2} & \dots & s_{2n-1} & x^n \end{bmatrix},$$

where  $s_k = x_1^k + x_2^k + \dots + x_n^k$ .

$$*296 \quad \begin{bmatrix} a & b & c & d & l & m & n & p \\ b & -a & -d & -c & m & -l & p & -n \\ c & d & -a & -b & n & -p & -l & m \\ d & -c & b & -a & p & n & -m & -l \\ l & -m & -n & -p & -a & b & c & d \\ m & l & p & -n & -b & -a & d & -c \\ n & -p & l & m & -c & -d & -a & b \\ p & n & -m & l & -d & c & -b & -a \end{bmatrix}.$$

$$*297 \quad \begin{bmatrix} \cos \varphi & \sin \varphi & \cos \varphi & \sin \varphi \\ \cos 2\varphi & \sin 2\varphi & 2 \cos 2\varphi & 2 \sin 2\varphi \\ \cos 3\varphi & \sin 3\varphi & 3 \cos 3\varphi & 3 \sin 3\varphi \\ \cos 4\varphi & \sin 4\varphi & 4 \cos 4\varphi & 4 \sin 4\varphi \end{bmatrix}.$$

$$*298 \quad \begin{bmatrix} \cos n\varphi & n \cos n\varphi & \sin n\varphi & n \sin n\varphi \\ \cos (n+1)\varphi & (n+1) \cos (n+1)\varphi & \sin (n+1)\varphi & (n+1) \sin (n+1)\varphi \\ \cos (n+2)\varphi & (n+2) \cos (n+2)\varphi & \sin (n+2)\varphi & (n+2) \sin (n+2)\varphi \\ \cos (n+3)\varphi & (n+3) \cos (n+3)\varphi & \sin (n+3)\varphi & (n+3) \sin (n+3)\varphi \end{bmatrix}.$$

$$*299 \quad \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \varepsilon & \varepsilon^2 & \dots & \varepsilon^{n-1} \\ 1 & \varepsilon^2 & \varepsilon^4 & \dots & \varepsilon^{2n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \varepsilon^{n-1} & \varepsilon^{2(n-1)} & \dots & \varepsilon^{(n-1)^2} \end{bmatrix},$$

$$\text{where } \varepsilon = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}.$$

\*300

$$\begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \dots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \dots & a_0 \end{bmatrix}.$$

(circulant matrix)

- 301 Use the result of problem 300 to calculate the determinant of the particular circulant

$$\begin{bmatrix} x & u & z & y \\ y & x & u & z \\ z & y & x & u \\ u & z & y & x \end{bmatrix}.$$

- 302 Also use the results of problem 300 to solve problems 192, 205, 255.

Calculate the determinants of the following matrices:

303 
$$\begin{bmatrix} 1 & C_1^{n-1} & C_2^{n-1} & \dots & C_{n-2}^{n-1} & 1 \\ 1 & 1 & C_1^{n-1} & \dots & C_{n-3}^{n-1} & C_{n-2}^{n-1} \\ C_{n-2}^{n-1} & 1 & 1 & \dots & C_{n-4}^{n-1} & C_{n-3}^{n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ C_1^{n-1} & C_2^{n-1} & C_3^{n-1} & \dots & 1 & 1 \end{bmatrix}.$$

304 
$$\begin{bmatrix} 1 & 2a & 3a^2 & \dots & na^{n-1} \\ na^{n-1} & 1 & 2a & \dots & (n-1)a^{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ 2a & 3a^2 & 4a^3 & \dots & 1 \end{bmatrix}.$$

305 
$$\begin{bmatrix} s-a_1 & s-a_2 & \dots & s-a_n \\ s-a_n & s-a_1 & \dots & s-a_{n-1} \\ \dots & \dots & \dots & \dots \\ s-a_2 & s-a_3 & \dots & s-a_1 \end{bmatrix},$$

where  $s = a_1 + a_2 + \dots + a_n$ .

306 
$$\begin{bmatrix} t^{n-1} & C_1^n t^{n-2} & C_2^n t^{n-3} & \dots & C_{n-2}^n t & C_{n-1}^n \\ C_{n-1}^n & t^{n-1} & C_1^n t^{n-2} & \dots & C_{n-3}^n t^2 & C_{n-2}^n t \\ C_{n-2}^n t & C_{n-1}^n & t^{n-1} & \dots & C_{n-4}^n t^3 & C_{n-3}^n t^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ C_1^n t^{n-2} & C_2^n t^{n-3} & C_3^n t^{n-4} & \dots & C_{n-1}^n & t^{n-1} \end{bmatrix}.$$

$$307 \quad \begin{bmatrix} \overbrace{-1 \ -1 \ \dots \ -1 \ -1}^p & \overbrace{1 \ 1 \ \dots \ 1}^{n-p} \\ 1 \ -1 \ \dots \ -1 \ -1 & -1 \ 1 \ \dots \ 1 \\ 1 \ 1 \ \dots \ -1 \ -1 & -1 \ -1 \ \dots \ 1 \\ \dots & \dots \\ -1 \ -1 \ \dots \ -1 \ 1 & 1 \ 1 \ \dots \ -1 \end{bmatrix}.$$

$$*308 \quad \begin{bmatrix} \cos \frac{\pi}{n} & \cos \frac{2\pi}{n} & \dots & -1 \\ -1 & \cos \frac{\pi}{n} & \dots & \cos \frac{(n-1)\pi}{n} \\ \cos \frac{(n-1)\pi}{n} & -1 & \dots & \cos \frac{(n-2)\pi}{n} \\ \dots & \dots & \dots & \dots \\ \cos \frac{2\pi}{n} & \cos \frac{3\pi}{n} & \dots & \cos \frac{\pi}{n} \end{bmatrix}.$$

$$309 \quad \begin{bmatrix} \cos \theta & \cos 2\theta & \dots & \cos n\theta \\ \cos n\theta & \cos \theta & \dots & \cos (n-1)\theta \\ \dots & \dots & \dots & \dots \\ \cos 2\theta & \cos 3\theta & \dots & \cos \theta \end{bmatrix}.$$

$$310 \quad \begin{bmatrix} \sin a & \sin (a+h) & \sin (a+2h) & \dots & \sin [a+(n-1)h] \\ \sin [a+(n-1)h] & \sin a & \sin (a+h) & \dots & \sin [a+(n-2)h] \\ \dots & \dots & \dots & \dots & \dots \\ \sin (a+h) & \sin (a+2h) & \sin (a+3h) & \dots & \sin a \end{bmatrix}.$$

$$*311 \quad \begin{bmatrix} 1^2 & 2^2 & 3^2 & \dots & n^2 \\ n^2 & 1^2 & 2^2 & \dots & (n-1)^2 \\ \dots & \dots & \dots & \dots & \dots \\ 2^2 & 3^2 & 4^2 & \dots & 1^2 \end{bmatrix}.$$

312 Establish the following relation

$$\det \begin{bmatrix} a_0 & a_1 & a_1 & a_2 & a_1 & a_2 & a_2 \\ a_2 & a_0 & a_1 & a_1 & a_2 & a_1 & a_2 \\ a_2 & a_2 & a_0 & a_1 & a_1 & a_2 & a_1 \\ a_1 & a_2 & a_2 & a_0 & a_1 & a_1 & a_2 \\ a_2 & a_1 & a_2 & a_2 & a_0 & a_1 & a_1 \\ a_1 & a_2 & a_1 & a_2 & a_2 & a_0 & a_1 \\ a_1 & a_1 & a_2 & a_1 & a_2 & a_2 & a_0 \end{bmatrix} =$$

$$= (a_0 + 3a_1 + 3a_2)(a_0^2 - a_0a_1 - a_0a_2 + 2a_1^2 + 2a_2^2 - 3a_1a_2)^3.$$

313 Calculate the determinant of the following skew-circulant matrix:

$$\begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ -a_n & a_1 & a_2 & \dots & a_{n-1} \\ -a_{n-1} & -a_n & a_1 & \dots & a_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ -a_2 & -a_3 & -a_4 & \dots & a_1 \end{bmatrix}$$

\*314 Let  $A$  be a circulant matrix of order  $2n$ . Show that  $\det A$  can be written as a product of  $\det B \cdot \det C$  where  $B$  is a circulant of order  $n$  and  $C$  is a skew-circulant of order  $n$ .

315 Calculate the determinant of the following matrix:

$$\begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ \mu a_n & a_1 & a_2 & \dots & a_{n-1} \\ \mu a_{n-1} & \mu a_n & a_1 & \dots & a_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ \mu a_2 & \mu a_3 & \mu a_4 & \dots & a_1 \end{bmatrix} \cdot$$

## 7. MISCELLANEOUS PROBLEMS

- 316 Show that the derivative of the determinant of the matrix

$$M = \begin{bmatrix} a_{11}(x) & a_{12}(x) & \dots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \dots & a_{2n}(x) \\ \dots & \dots & \dots & \dots \\ a_{n1}(x) & a_{n2}(x) & \dots & a_{nn}(x) \end{bmatrix},$$

is given by the formula

$$\begin{aligned} (\det M)' = \Delta'(x) = \det \begin{bmatrix} a'_{11}(x) & a'_{12}(x) & \dots & a'_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \dots & a_{2n}(x) \\ \dots & \dots & \dots & \dots \\ a_{n1}(x) & a_{n2}(x) & \dots & a_{nn}(x) \end{bmatrix} + \dots \\ \dots + \det \begin{bmatrix} a_{11}(x) & a_{12}(x) & \dots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \dots & a_{2n}(x) \\ \dots & \dots & \dots & \dots \\ a'_{n1}(x) & a'_{n2}(x) & \dots & a'_{nn}(x) \end{bmatrix}. \end{aligned}$$

- 317 Let  $A_{ik}$  be the algebraic co-factors of the elements  $a_{ik}$  in the matrix  $[a_{ik}]$ . Establish the relation

$$\begin{aligned} \det \begin{bmatrix} a_{11} + x & a_{12} + x & \dots & a_{1n} + x \\ a_{21} + x & a_{22} + x & \dots & a_{2n} + x \\ \dots & \dots & \dots & \dots \\ a_{n1} + x & a_{n2} + x & \dots & a_{nn} + x \end{bmatrix} &= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} + \\ &+ x \sum_{k=1}^n \sum_{i=1}^n A_{ik}, \end{aligned}$$

- 318 Show how the result of problem 317 can be used to compute the results of problems 200, 223, 224, 225, 226, 227, 228, 232, 233, 248, 249, and 250.
- 319 Show that the sum of the algebraic co-factors of all the elements of the square matrix  $[a_{ij}]$  has the value

$$\det \begin{bmatrix} 1 & 1 & \dots & 1 \\ a_{21} - a_{11} & a_{22} - a_{12} & \dots & a_{2n} - a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{n1} - a_{n-1,1} & a_{n2} - a_{n-1,2} & \dots & a_{nn} - a_{n-1,n} \end{bmatrix}.$$

Establish the following theorems:

- 320 Let the matrix  $B$  be obtained from the matrix  $A$  by adding one and the same constant to every element of  $A$ . Show that the sum of the co-factors of the elements of  $A$  is the same as the sum of the co-factors of the elements of  $B$ .
- 321 Suppose all the elements of some row [column] of a matrix are unity. Show that the sum of the algebraic co-factors of the elements of the matrix is equal to the determinant of the matrix.
- 322 In problem 250 find the sum of all the algebraic co-factors of the elements of the matrix.



\*323 Calculate the determinant of the matrix

$$\begin{bmatrix} (a_1 + b_1)^{-1} & (a_1 + b_2)^{-1} & \dots & (a_1 + b_n)^{-1} \\ (a_2 + b_1)^{-1} & (a_2 + b_2)^{-1} & \dots & (a_2 + b_n)^{-1} \\ \dots & \dots & \dots & \dots \\ (a_n + b_1)^{-1} & (a_n + b_2)^{-1} & \dots & (a_n + b_n)^{-1} \end{bmatrix}.$$

324 Let  $P_n, Q_n$  represent the following two determinants

$$\det \begin{bmatrix} a_0 & 1 & 0 & \dots & 0 & 0 \\ -1 & a_1 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{n-2} & 1 \\ 0 & 0 & 0 & \dots & -1 & a_{n-1} \end{bmatrix}, \det \begin{bmatrix} a_1 & 1 & 0 & \dots & 0 & 0 \\ -1 & a_2 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{n-2} & 1 \\ 0 & 0 & 0 & \dots & -1 & a_{n-1} \end{bmatrix}$$

Establish the relation

$$\frac{P_n}{Q_n} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots + \frac{1}{a_{n-1}}}}}.$$

Calculate the determinants of the following matrices:

$$\begin{array}{ll}
 *325 & \begin{bmatrix} c & a & 0 & \dots & 0 & 0 \\ b & c & a & \dots & 0 & 0 \\ 0 & b & c & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & c & a \\ 0 & 0 & 0 & \dots & b & c \end{bmatrix} \\
 326 & \begin{bmatrix} p & q & 0 & \dots & 0 & 0 \\ 2 & p & q & \dots & 0 & 0 \\ 0 & 1 & p & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & p & q \\ 0 & 0 & 0 & \dots & 1 & p \end{bmatrix}
 \end{array}$$

- \*327 Write as a polynomial in  $x$  the determinant of the following matrix

$$\begin{bmatrix} a_{11} + x & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} + x & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} + x \end{bmatrix}$$

- \*328 Calculate the determinate of the matrix of order  $2n - 1$  that has the following form. The first  $n - 1$  elements of the principal diagonal are unity, the remaining elements of the principal diagonal are  $n$ . In the first  $n - 1$  rows of the matrix there is a sequence of  $n$  1's. In the remaining  $n$  rows of the matrix the elements to the left of the principal diagonal, reading from the principal diagonal leftward, are  $n - 1, n - 2, \dots, 1$ . All other elements of the matrix are 0. For example,

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}; \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 \end{bmatrix}.$$

Calculate the determinant of the following matrix:

\*329

$$\begin{bmatrix} x & 1 & 0 & 0 & \dots & 0 & 0 \\ -n & x-2 & 2 & 0 & \dots & 0 & 0 \\ 0 & -(n-1) & x-4 & 3 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & -1, x-2n \end{bmatrix}.$$

330

$$\begin{bmatrix} x & 1 & 0 & 0 & \dots & 0 & 0 \\ n-1 & x & 2 & 0 & \dots & 0 & 0 \\ 0 & n-2 & x & 3 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 1 & x \end{bmatrix}.$$

$$331 \quad \begin{bmatrix} x & a & 0 & 0 & \dots & 0 & 0 \\ n(a-1) & x-1 & 2a & 0 & \dots & 0 & 0 \\ 0 & (n-1)(a-1) & x-2 & 3a & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & a-1 & x-n \end{bmatrix}.$$

$$332 \quad \begin{bmatrix} 1^{n-1} & 2^{n-1} & \dots & n^{n-1} \\ 2^{n-1} & 3^{n-1} & \dots & (n+1)^{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ n^{n-1} & (n+1)^{n-1} & \dots & (2n-1)^{n-1} \end{bmatrix}.$$

$$333 \quad \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \dots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \dots & \frac{1}{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \dots & \frac{1}{2n-1} \end{bmatrix}.$$

- 334 Calculate the coefficient of the highest power of  $x$  in the determinant of the matrix

$$\begin{bmatrix} (1+x)^{a_1 b_1} & (1+x)^{a_1 b_2} & \dots & (1+x)^{a_1 b_n} \\ (1+x)^{a_2 b_1} & (1+x)^{a_2 b_2} & \dots & (1+x)^{a_2 b_n} \\ \cdot & \cdot & \cdot & \cdot \\ (1+x)^{a_n b_1} & (1+x)^{a_n b_2} & \dots & (1+x)^{a_n b_n} \end{bmatrix}.$$

# CHAPTER III - PROBLEMS

## SYSTEMS OF LINEAR EQUATIONS

### 1. CRAMER'S RULE

Solve each of the following systems of equations:

$$\begin{aligned} 335 \quad & 2x_1 - x_2 - x_3 = 4, \\ & 3x_1 + 4x_2 - 2x_3 = 11, \\ & 3x_1 - 2x_2 + 4x_3 = 11. \end{aligned}$$

$$\begin{aligned} 337 \quad & 3x_1 + 2x_2 + x_3 = 5, \\ & 2x_1 + 3x_2 + x_3 = 1, \\ & 2x_1 + x_2 + 3x_3 = 11. \end{aligned}$$

$$\begin{aligned} 339 \quad & x_1 + x_2 + 2x_3 + 3x_4 = 1, \\ & 3x_1 - x_2 - x_3 - 2x_4 = -4, \\ & 2x_1 + 3x_2 - x_3 - x_4 = -6, \\ & x_1 + 2x_2 + 3x_3 - x_4 = -4. \end{aligned}$$

$$\begin{aligned} 341 \quad & x_1 + 2x_2 + 3x_3 + 4x_4 = 5, \\ & 2x_1 + x_2 + 2x_3 + 3x_4 = 1, \\ & 3x_1 + 2x_2 + x_3 + 2x_4 = 1, \\ & 4x_1 + 3x_2 + 2x_3 + x_4 = -5. \end{aligned}$$

$$\begin{aligned} 343 \quad & 2x_1 - x_2 + 3x_3 + 2x_4 = 4, \\ & 3x_1 + 3x_2 + 3x_3 + 2x_4 = 6, \\ & 3x_1 - x_2 - x_3 + 2x_4 = 6, \\ & 3x_1 - x_2 + 3x_3 - x_4 = 6. \end{aligned}$$

$$\begin{aligned} 336 \quad & x_1 + x_2 + 2x_3 = -1, \\ & 2x_1 - x_2 + 2x_3 = -4, \\ & 4x_1 + x_2 + 4x_3 = -2. \end{aligned}$$

$$\begin{aligned} 338 \quad & x_1 + 2x_2 + 4x_3 = 31, \\ & 5x_1 + x_2 + 2x_3 = 29, \\ & 3x_1 - x_2 + x_3 = 10. \end{aligned}$$

$$\begin{aligned} 340 \quad & x_1 + 2x_2 + 3x_3 - 2x_4 = 6, \\ & 2x_1 - x_2 - 2x_3 - 3x_4 = 8, \\ & 3x_1 + 2x_2 - x_3 + 2x_4 = 4, \\ & 2x_1 - 3x_2 + 2x_3 + x_4 = -8. \end{aligned}$$

$$\begin{aligned} 342 \quad & x_2 - 3x_3 + 4x_4 = -5, \\ & x_1 - 2x_3 + 3x_4 = -4, \\ & 3x_1 + 2x_2 - 5x_4 = 12, \\ & 4x_1 + 3x_2 - 5x_3 = 5. \end{aligned}$$

$$\begin{aligned} 344 \quad & x_1 + x_2 + x_3 + x_4 = 0, \\ & x_1 + 2x_2 + 3x_3 + 4x_4 = 0, \\ & x_1 + 3x_2 + 6x_3 + 10x_4 = 0, \\ & x_1 + 4x_2 + 10x_3 + 20x_4 = 0. \end{aligned}$$

345

$$\begin{aligned} x_1 + 3x_2 + 5x_3 + 7x_4 &= 12, & x_1 + x_2 + x_3 + x_4 + x_5 &= 0, \\ 3x_1 + 5x_2 + 7x_3 + x_4 &= 0, & x_1 - x_2 + 2x_3 - 2x_4 + 3x_5 &= 0, \\ 5x_1 + 7x_2 + x_3 + 3x_4 &= 4, & x_1 + x_2 + 4x_3 + 4x_4 + 9x_5 &= 0, \\ 7x_1 + x_2 + 3x_3 + 5x_4 &= 16, & x_1 - x_2 + 8x_3 - 8x_4 + 27x_5 &= 0, \\ & & x_1 + x_2 + 16x_3 + 16x_4 + 81x_5 &= 0. \end{aligned}$$

346

347

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 0, & x_1 + x_2 + x_3 + x_4 &= 0, \\ x_1 + x_2 + 2x_3 + 3x_4 &= 0, & x_2 + x_3 + x_4 + x_5 &= 0, \\ x_1 + 5x_2 + x_3 + 2x_4 &= 0, & x_1 + 2x_2 + 3x_3 &= 2, \\ x_1 + 5x_2 + 5x_3 + 2x_4 &= 0, & x_2 + 2x_3 + 3x_4 &= -2, \\ & & x_3 + 2x_4 + 3x_5 &= 2. \end{aligned}$$

348

349

$$\begin{aligned} x_1 + 4x_2 + 6x_3 + 4x_4 + x_5 &= 0, \\ x_1 + x_2 + 4x_3 + 6x_4 + 4x_5 &= 0, \\ 4x_1 + x_2 + x_3 + 4x_4 + 6x_5 &= 0, \\ 6x_1 + 4x_2 + x_3 + x_4 + 4x_5 &= 0, \\ 4x_1 + 6x_2 + 4x_3 + x_4 + x_5 &= 0. \end{aligned}$$

350

$$\begin{aligned} 2x_1 + x_2 + x_3 + x_4 + x_5 &= 2, \\ x_1 + 2x_2 + x_3 + x_4 + x_5 &= 0, \\ x_1 + x_2 + 3x_3 + x_4 + x_5 &= 3, \\ x_1 + x_2 + x_3 + 4x_4 + x_5 &= -2, \\ x_1 + x_2 + x_3 + x_4 + 5x_5 &= 5. \end{aligned}$$

351

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 &= 13, \\ 2x_1 + x_2 + 2x_3 + 3x_4 + 4x_5 &= 10, \\ 2x_1 + 2x_2 + x_3 + 2x_4 + 3x_5 &= 11, \\ 2x_1 + 2x_2 + 2x_3 + x_4 + 2x_5 &= 6, \\ 2x_1 + 2x_2 + 2x_3 + 2x_4 + x_5 &= 3. \end{aligned}$$

352

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + 4x_4 - x_5 &= -1, \\ 2x_1 - x_2 + 3x_3 - 4x_4 + 2x_5 &= 8, \\ 3x_1 + x_2 - x_3 + 2x_4 - x_5 &= 3, \\ 4x_1 + 3x_2 + 4x_3 + 2x_4 + 2x_5 &= -2, \\ x_1 - x_2 - x_3 + 2x_4 - 3x_5 &= -3. \end{aligned}$$



- 356 If the numbers  $b_1, b_2, \dots, b_n, \beta_1, \beta_2, \dots, \beta_n$  are  $2n$  different numbers, find a solution of the following system:

$$\begin{aligned} \frac{x_1}{b_1 - \beta_1} + \frac{x_2}{b_1 - \beta_2} + \dots + \frac{x_n}{b_1 - \beta_n} &= 1, \\ \frac{x_1}{b_2 - \beta_1} + \frac{x_2}{b_2 - \beta_2} + \dots + \frac{x_n}{b_2 - \beta_n} &= 1, \\ \dots & \\ \frac{x_1}{b_n - \beta_1} + \frac{x_2}{b_n - \beta_2} + \dots + \frac{x_n}{b_n - \beta_n} &= 1, \end{aligned}$$

- 357 If  $\alpha_1, \alpha_2, \dots, \alpha_n$  are all different, find a solution of the following system:

$$\begin{aligned} x_1 + x_2 + \dots + x_n &= 1, \\ x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n &= t, \\ \dots & \\ x_1\alpha_1^{n-1} + x_2\alpha_2^{n-1} + \dots + x_n\alpha_n^{n-1} &= t^{n-1}, \end{aligned}$$

- 358 If  $\alpha_1, \alpha_2, \dots, \alpha_n$  are all different, find a solution of the following system:

$$\begin{aligned} x_1 + x_2\alpha_1 + \dots + x_n\alpha_1^{n-1} &= u_1, \\ x_1 + x_2\alpha_2 + \dots + x_n\alpha_2^{n-1} &= u_2, \\ \dots & \\ x_1 + x_2\alpha_n + \dots + x_n\alpha_n^{n-1} &= u_n. \end{aligned}$$



- 359 If  $\alpha_1, \alpha_2, \dots, \alpha_n$  are all different, find a solution of the following system:

$$\begin{aligned} x_1 + x_2 + \dots + x_n &= u_1, \\ x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n &= u_2, \\ \dots &\dots \\ x_1\alpha_1^{n-1} + x_2\alpha_2^{n-1} + \dots + x_n\alpha_n^{n-1} &= u_n. \end{aligned}$$

- 360 Solve the following system of equations:

$$\begin{aligned} 1 + x_1 + x_2 + \dots + x_n &= 0, \\ 1 + 2x_1 + 2^2x_2 + \dots + 2^nx_n &= 0, \\ \dots &\dots \\ 1 + nx_1 + n^2x_2 + \dots + n^nx_n &= 0. \end{aligned}$$

## 2. THE RANK OF A MATRIX

- 361 How many  $k$ -th order submatrices are there in a matrix of  $m$  rows and  $n$  columns?
- 362 Construct matrices of rank a) 2; b) 3.
- 363 Show that the following operations do not change the rank of a matrix:
- interchange of rows and columns;
  - multiplication of all the elements of a row [column] by a non-zero quantity;
  - interchange of two rows [columns];
  - addition to the elements of one row [column] the corresponding elements of another row [column] multiplied by an arbitrary quantity.

- 364 If two matrices  $A, B$  have the same number of rows and columns, the matrix obtained by adding corresponding elements of  $A$  and  $B$  is called the sum of the two matrices:  $C = A + B$ . Show that the rank of the sum of two matrices cannot exceed the sum of the ranks of the individual matrices.
- 365 How is the rank of the matrix altered if it is bordered by a) 1 additional column; b) 2 additional columns?

Calculate the rank of each of the following matrices:

- 366  $\begin{pmatrix} 0 & 4 & 10 & 1 \\ 4 & 8 & 18 & 7 \\ 10 & 18 & 40 & 17 \\ 1 & 7 & 17 & 3 \end{pmatrix}$ .
- 367  $\begin{pmatrix} 75 & 0 & 116 & -39 & 0 \\ 171 & -69 & 402 & 123 & 45 \\ 301 & 0 & 87 & -417 & -169 \\ 114 & -46 & 268 & 82 & 30 \end{pmatrix}$ .
- 368  $\begin{pmatrix} 2 & 1 & 11 & 2 \\ 1 & 0 & 4 & -1 \\ 11 & 4 & 56 & 5 \\ 2 & -1 & 5 & -6 \end{pmatrix}$ .
- 369  $\begin{pmatrix} 14 & 12 & 6 & 8 & 2 \\ 6 & 104 & 21 & 9 & 17 \\ 7 & 6 & 3 & 4 & 1 \\ 35 & 30 & 15 & 20 & 5 \end{pmatrix}$ .
- 370  $\begin{pmatrix} 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 3 & 6 \\ 1 & 2 & 3 & 14 & 32 \\ 4 & 5 & 6 & 32 & 77 \end{pmatrix}$ .
- 371  $\begin{pmatrix} 1 & -2 & 3 & -1 & -1 & -2 \\ 2 & -1 & 1 & 0 & -2 & -2 \\ -2 & -5 & 8 & -4 & 3 & -1 \\ 6 & 0 & -1 & 2 & -7 & -5 \\ -1 & -1 & 1 & -1 & 2 & 1 \end{pmatrix}$ .
- 372  $\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 5 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ .
- 373  $\begin{pmatrix} 1 & -1 & 2 & 3 & 4 \\ 2 & 1 & -1 & 2 & 0 \\ -1 & 2 & 1 & 1 & 3 \\ 1 & 5 & -8 & -5 & -12 \\ 3 & -7 & 8 & 9 & 13 \end{pmatrix}$ .
- 374  $\begin{pmatrix} 2 & 1 & 3 & -1 \\ 3 & -1 & 2 & 0 \\ 1 & 3 & 4 & -2 \\ 4 & -3 & 1 & 1 \end{pmatrix}$ .

$$375 \begin{pmatrix} 3 & 2 & -1 & 2 & 0 & 1 \\ 4 & 1 & 0 & -3 & 0 & 2 \\ 2 & -1 & -2 & 1 & 1 & -3 \\ 3 & 1 & 3 & -9 & -1 & 6 \\ 3 & -1 & -5 & 7 & 2 & -7 \end{pmatrix}.$$

$$376 \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 4 & 5 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \end{pmatrix}.$$

$$377 \begin{pmatrix} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 1 & -1 & 2 & 0 & 1 \\ 1 & 0 & -1 & 0 & 2 & 1 \\ 1 & -1 & 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 1 & -1 & 1 \\ -1 & 1 & 0 & 1 & 1 & 2 \end{pmatrix}.$$

$$378 \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

$$379 \begin{pmatrix} 2 & 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

$$380 \begin{pmatrix} 2 & -1 & 1 & 3 & 4 \\ 2 & -1 & 2 & 1 & -2 \\ 2 & -3 & 1 & 2 & -2 \\ 1 & 0 & 1 & -2 & -6 \\ 1 & 2 & 1 & -1 & 0 \\ 4 & -1 & 3 & -1 & -8 \end{pmatrix}.$$

## 3. SYSTEMS OF LINEAR FORMS

- 381 a) Give an example of two independent linear forms;  
 b) Give an example of three independent linear forms.
- 382 Give a system of four linear forms in five indeterminates, such that two of the forms are independent and the remaining are linear combinations of these two.

Find fundamental linear relations among the following systems of forms:

383

$$\begin{aligned} y_1 &= 2x_1 + 2x_2 + 7x_3 - x_4, \\ y_2 &= 3x_1 - x_2 + 2x_3 + 4x_4, \\ y_3 &= x_1 + x_2 + 3x_3 + x_4. \end{aligned}$$

384

$$\begin{aligned} y_1 &= 3x_1 + 2x_2 - 5x_3 + 4x_4, \\ y_2 &= 3x_1 - x_2 + 3x_3 - 3x_4, \\ y_3 &= 3x_1 + 5x_2 - 13x_3 + 11x_4. \end{aligned}$$

385

$$\begin{aligned} y_1 &= 2x_1 + 3x_2 - 4x_3 - x_4, \\ y_2 &= x_1 - 2x_2 + x_3 + 3x_4, \\ y_3 &= 5x_1 - 3x_2 - x_3 + 8x_4, \\ y_4 &= 3x_1 + 8x_2 - 9x_3 - 5x_4. \end{aligned}$$

386

$$\begin{aligned} y_1 &= 2x_1 + x_2 - x_3 + x_4, \\ y_2 &= x_1 + 2x_2 + x_3 - x_4, \\ y_3 &= x_1 + x_2 + 2x_3 + x_4. \end{aligned}$$

387

$$\begin{aligned} y_1 &= x_1 + 2x_2 + 3x_3 + x_4, \\ y_2 &= 2x_1 + 3x_2 + x_3 + 2x_4, \\ y_3 &= 3x_1 + x_2 + 2x_3 - 2x_4, \\ y_4 &= 4x_2 + 2x_3 + 5x_4. \end{aligned}$$

388

$$\begin{aligned} y_1 &= 2x_1 + x_2, \\ y_2 &= 3x_1 + 2x_2, \\ y_3 &= x_1 + x_2, \\ y_4 &= 2x_1 + 3x_2. \end{aligned}$$

389

$$\begin{aligned} y_1 &= x_1 + x_2 + x_3 + x_4 + x_5, \\ y_2 &= x_1 + 2x_2 + 3x_3 + 4x_4 + x_5, \\ y_3 &= x_1 + 3x_2 + 6x_3 + 10x_4 + x_5, \\ y_4 &= x_1 + 4x_2 + 10x_3 + 20x_4 + x_5. \end{aligned}$$

390

$$\begin{aligned} y_1 &= x_1 + 2x_2 + 3x_3 - 4x_4, \\ y_2 &= 2x_1 - x_2 + 2x_3 + 5x_4, \\ y_3 &= 2x_1 - x_2 + 5x_3 - 4x_4, \\ y_4 &= 2x_1 + 3x_2 - 4x_3 + x_4. \end{aligned}$$

391

$$\begin{aligned} y_1 &= 2x_1 + x_2 - 3x_3, \\ y_2 &= 3x_1 + x_2 - 5x_3, \\ y_3 &= 4x_1 + 2x_2 - x_3, \\ y_4 &= x_1 - 7x_3. \end{aligned}$$

$$\begin{aligned} 392 \quad y_1 &= 2x_1 + 3x_2 + 5x_3 - 4x_4 + x_5, \\ y_2 &= x_1 - x_2 + 2x_3 + 3x_4 + 5x_5, \\ y_3 &= 3x_1 + 7x_2 + 8x_3 - 11x_4 - 3x_5, \\ y_4 &= x_1 - x_2 + x_3 - 2x_4 + 3x_5. \end{aligned}$$

$$\begin{aligned} 393 \quad y_1 &= 2x_1 - x_2 + 3x_3 + 4x_4 - x_5, \\ y_2 &= x_1 + 2x_2 - 3x_3 + x_4 + 2x_5, \\ y_3 &= 5x_1 - 5x_2 + 12x_3 + 11x_4 - 5x_5, \\ y_4 &= x_1 - 3x_2 + 6x_3 + 3x_4 - 3x_5. \end{aligned}$$

$$\begin{aligned} 394 \quad y_1 &= x_1 + 2x_2 + x_3 - 2x_4 + x_5, \\ y_2 &= 2x_1 - x_2 + x_3 + 3x_4 + 2x_5, \\ y_3 &= x_1 - x_2 + 2x_3 - x_4 + 3x_5, \\ y_4 &= 2x_1 + x_2 - 3x_3 + x_4 - 2x_5, \\ y_5 &= x_1 - x_2 + 3x_3 - x_4 + 7x_5. \end{aligned}$$

$$\begin{aligned} 395 \quad y_1 &= 4x_1 + 3x_2 - x_3 + x_4 - x_5, \\ y_2 &= 2x_1 + x_2 - 3x_3 + 2x_4 - 5x_5, \\ y_3 &= x_1 - 3x_2 + x_4 - 2x_5, \\ y_4 &= x_1 + 5x_2 + 2x_3 - 2x_4 + 6x_5. \end{aligned}$$

$$\begin{aligned} 396 \quad y_1 &= x_1 + 2x_2 - x_3 + 3x_4 - x_5 + 2x_6, \\ y_2 &= 2x_1 - x_2 + 3x_3 - 4x_4 + x_5 - x_6, \\ y_3 &= 3x_1 + x_2 - x_3 + 2x_4 + x_5 + 3x_6, \\ y_4 &= 4x_1 - 7x_2 + 8x_3 - 15x_4 + 6x_5 - 5x_6, \\ y_5 &= 5x_1 + 5x_2 - 6x_3 + 11x_4 + 9x_6. \end{aligned}$$

- 397 In the following system, find what value  $\lambda$  must have so that the fourth form is a linear combination of the first three.

$$\begin{aligned}y_1 &= x_1 + 2x_2 + x_3 - 3x_4 + 2x_5, \\y_2 &= 2x_1 + x_2 + x_3 + x_4 - 3x_5, \\y_3 &= x_1 + x_2 + 2x_3 + 2x_4 - 2x_5, \\y_4 &= 2x_1 + 3x_2 - 5x_3 - 17x_4 + \lambda x_5.\end{aligned}$$

#### 4. SYSTEMS OF LINEAR EQUATIONS

- 398 Solve the following system of equations:

$$\begin{aligned}x_1 - 2x_2 + x_3 + x_4 &= 1, \\x_1 - 2x_2 + x_3 - x_4 &= -1, \\x_1 - 2x_2 + x_3 + 5x_4 &= 5.\end{aligned}$$

- 399 For what value of  $\lambda$  does the following system of equations have a solution:

$$\begin{aligned}2x_1 - x_2 + x_3 + x_4 &= 1, \\x_1 + 2x_2 - x_3 + 4x_4 &= 2, \\x_1 + 7x_2 - 4x_3 + 11x_4 &= \lambda.\end{aligned}$$

Solve the following systems of equations:

400  $x_1 + x_2 - 3x_3 = -1,$   
 $2x_1 + x_2 - 2x_3 = 1,$   
 $x_1 + x_2 + x_3 = 3,$   
 $x_1 + 2x_2 - 3x_3 = 1.$

401  $2x_1 + x_2 + x_3 = 2,$   
 $x_1 + 3x_2 + x_3 = 5,$   
 $x_1 + x_2 + 5x_3 = -7,$   
 $2x_1 + 3x_2 - 3x_3 = 14.$

$$402 \quad 2x_1 - x_2 + 3x_3 = 3,$$

$$3x_1 + x_2 - 5x_3 = 0,$$

$$4x_1 - x_2 + x_3 = 3,$$

$$x_1 + 3x_2 - 13x_3 = -6.$$

$$403 \quad x_1 + 3x_2 + 2x_3 = 0,$$

$$2x_1 - x_2 + 3x_3 = 0,$$

$$3x_1 - 5x_2 + 4x_3 = 0,$$

$$x_1 + 17x_2 + 4x_3 = 0.$$

404

$$2x_1 + x_2 - x_3 + x_4 = 1,$$

$$3x_1 - 2x_2 + 2x_3 - 3x_4 = 2,$$

$$5x_1 + x_2 - x_3 + 2x_4 = -1,$$

$$2x_1 - x_2 + x_3 - 3x_4 = 4,$$

405

$$2x_1 - x_2 + x_3 - x_4 = 1,$$

$$2x_1 - x_2 - 3x_4 = 2,$$

$$3x_1 - x_3 + x_4 = -3,$$

$$2x_1 + 2x_2 - 2x_3 + 5x_4 = -6.$$

406

$$x_1 - 2x_2 + 3x_3 - 4x_4 = 4,$$

$$x_2 - x_3 + x_4 = -3,$$

$$x_1 + 3x_2 - 3x_4 = 1,$$

$$-7x_2 + 3x_3 + x_4 = -3.$$

407

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 11,$$

$$2x_1 + 3x_2 + 4x_3 + x_4 = 12,$$

$$3x_1 + 4x_2 + x_3 + 2x_4 = 13,$$

$$4x_1 + x_2 + 2x_3 + 3x_4 = 14.$$

408

$$2x_1 + 3x_2 - x_3 + 5x_4 = 0,$$

$$3x_1 - x_2 + 2x_3 - 7x_4 = 0,$$

$$4x_1 + x_2 - 3x_3 + 6x_4 = 0,$$

$$x_1 - 2x_2 + 4x_3 - 7x_4 = 0.$$

409

$$3x_1 + 4x_2 - 5x_3 + 7x_4 = 0,$$

$$2x_1 - 3x_2 + 3x_3 - 2x_4 = 0,$$

$$4x_1 + 11x_2 - 13x_3 + 16x_4 = 0,$$

$$7x_1 - 2x_2 + x_3 + 3x_4 = 0.$$

410

$$x_1 + x_2 - 3x_4 - x_5 = 0,$$

$$x_1 - x_2 + 2x_3 - x_4 = 0,$$

$$4x_1 - 2x_2 + 6x_3 + 3x_4 - 4x_5 = 0,$$

$$2x_1 + 4x_2 - 2x_3 + 4x_4 - 7x_5 = 0.$$

411

$$x_1 + x_2 + x_3 + x_4 + x_5 = 7,$$

$$3x_1 + 2x_2 + x_3 + x_4 - 3x_5 = -2,$$

$$x_2 + 2x_3 + 2x_4 + 6x_5 = 23,$$

$$5x_1 + 4x_2 + 3x_3 + 3x_4 - x_5 = 12.$$

412

$$x_1 - 2x_2 + x_3 - x_4 + x_5 = 0,$$

$$2x_1 + x_2 - x_3 + 2x_4 - 3x_5 = 0,$$

$$3x_1 - 2x_2 - x_3 + x_4 - 2x_5 = 0,$$

$$2x_1 - 5x_2 + x_3 - 2x_4 + 2x_5 = 0.$$

$$\begin{aligned}
 413 \quad & x_1 - 2x_2 + x_3 + x_4 - x_5 = 0, \\
 & 2x_1 + x_2 - x_3 - x_4 + x_5 = 0, \\
 & x_1 + 7x_2 - 5x_3 - 5x_4 + 5x_5 = 0, \\
 & 3x_1 - x_2 - 2x_3 + x_4 - x_5 = 0.
 \end{aligned}$$

$$\begin{aligned}
 414 \quad & 2x_1 + x_2 - x_3 - x_4 + x_5 = 1, \\
 & x_1 - x_2 + x_3 + x_4 - 2x_5 = 0, \\
 & 3x_1 + 3x_2 - 3x_3 - 3x_4 + 4x_5 = 2, \\
 & 4x_1 + 5x_2 - 5x_3 - 5x_4 + 7x_5 = 3.
 \end{aligned}$$

$$\begin{aligned}
 415 \quad & 2x_1 - 2x_2 + x_3 - x_4 + x_5 = 1, \\
 & x_1 + 2x_2 - x_3 + x_4 - 2x_5 = 1, \\
 & 4x_1 - 10x_2 + 5x_3 - 5x_4 + 7x_5 = 1, \\
 & 2x_1 - 14x_2 + 7x_3 - 7x_4 + 11x_5 = -1.
 \end{aligned}$$

$$\begin{aligned}
 416 \quad & 3x_1 + x_2 - 2x_3 + x_4 - x_5 = 1, \\
 & 2x_1 - x_2 + 7x_3 - 3x_4 + 5x_5 = 2, \\
 & x_1 + 3x_2 - 2x_3 + 5x_4 - 7x_5 = 3, \\
 & 3x_1 - 2x_2 + 7x_3 - 5x_4 + 8x_5 = 3.
 \end{aligned}$$

$$\begin{aligned}
 417 \quad & x_1 + 2x_2 - 3x_4 + 2x_5 = 1, \\
 & x_1 - x_2 - 3x_3 + x_4 - 3x_5 = 2, \\
 & 2x_1 - 3x_2 + 4x_3 - 5x_4 + 2x_5 = 7, \\
 & 9x_1 - 9x_2 + 6x_3 - 16x_4 + 2x_5 = 25.
 \end{aligned}$$

418

$$\begin{aligned}
 x_1 + 3x_2 + 5x_3 - 4x_4 &= 1, \\
 x_1 + 3x_2 + 2x_3 - 2x_4 + x_5 &= -1, \\
 x_1 - 2x_2 + x_3 - x_4 - x_5 &= 3, \\
 x_1 - 4x_2 + x_3 + x_4 - x_5 &= 3, \\
 x_1 + 2x_2 + x_3 - x_4 + x_5 &= -1,
 \end{aligned}$$

419

$$\begin{aligned}
 x_1 + 2x_2 + 3x_3 - x_4 &= 1, \\
 3x_1 + 2x_2 + x_3 - x_4 &= 1, \\
 2x_1 + 3x_2 + x_3 + x_4 &= 1, \\
 2x_1 + 2x_2 + 2x_3 - x_4 &= 1, \\
 5x_1 + 5x_2 + 2x_3 &= 2.
 \end{aligned}$$



420

$$\begin{aligned}
 x_1 - 2x_2 + 3x_3 - 4x_4 + 2x_5 &= -2, \\
 x_1 + 2x_2 - x_3 - x_5 &= -3, \\
 x_1 - x_2 + 2x_3 - 3x_4 &= 10, \\
 x_2 - x_3 + x_4 - 2x_5 &= -5, \\
 2x_1 + 3x_2 - x_3 + x_4 + 4x_5 &= 1.
 \end{aligned}$$

421 If it is known that the system of equations

$$\begin{aligned}
 ay + bx &= c, \\
 cx + az &= b, \\
 bz + cy &= a
 \end{aligned}$$

has a single solution, show that  $abc \neq 0$  and find the solution itself.

Solve the following systems of equations:

- |  |   |
|--|---|
| <p>422 <math>\lambda x + y + z = 1,</math><br/> <math>x + \lambda y + z = \lambda,</math><br/> <math>x + y + \lambda z = \lambda^2.</math></p> | <p>423 <math>\lambda x + y + z + t = 1,</math><br/> <math>x + \lambda y + z + t = \lambda,</math><br/> <math>x + y + \lambda z + t = \lambda^2,</math><br/> <math>x + y + z + \lambda t = \lambda^3.</math></p> |
| <p>424 <math>x + ay + a^2z = a^3,</math><br/> <math>x + by + b^2z = b^3,</math><br/> <math>x + cy + c^2z = c^3.</math></p>                     | <p>425 <math>x + y + z = 1,</math><br/> <math>ax + by + cz = d,</math><br/> <math>a^2x + b^2y + c^2z = d^2.</math></p>  |
| <p>426 <math>ax + y + z = 4,</math><br/> <math>x + by + z = 3,</math><br/> <math>x + 2by + z = 4.</math></p>                                   | <p>427 <math>ax + by + z = 1,</math><br/> <math>x + aby + z = b,</math><br/> <math>x + by + az = 1.</math></p>  |
| <p>428 <math>ax + y + z = m,</math><br/> <math>x + ay + z = n,</math><br/> <math>x + y + az = p.</math></p>                                    | <p>429 <math>x + ay + a^2z = 1,</math><br/> <math>x + ay + abz = a,</math><br/> <math>bx + a^2y + a^2bz = a^2b.</math></p>  |

430

$$\begin{aligned}
 (\lambda+3)x + y + 2z &= \lambda, & \lambda x + \lambda y + (\lambda+1)z &= \lambda, \\
 \lambda x + (\lambda-1)y + z &= 2\lambda, & \lambda x + \lambda y + (\lambda-1)z &= \lambda, \\
 3(\lambda+1)x + \lambda y + (\lambda+3)z &= 3. & (\lambda+1)x + \lambda y + (2\lambda+3)z &= 1.
 \end{aligned}$$

431

$$\begin{aligned}
 432 \quad 3kx + (2k+1)y + (k+1)z &= k, \\
 (2k-1)x + (2k-1)y + (k-2)z &= k+1, \\
 (4k-1)x + 3ky + 2kz &= 1.
 \end{aligned}$$

$$\begin{aligned}
 433 \quad ax + by + 2z &= 1, \\
 ax + (2b-1)y + 3z &= 1, \\
 ax + by + (b+3)z &= 2b-1.
 \end{aligned}$$

$$\begin{aligned}
 434 \quad a) \quad 3mx + (3m-7)y + (m-5)z &= m-1, \\
 (2m-1)x + (4m-1)y + 2mz &= m+1, \\
 4mx + (5m-7)y + (2m-5)z &= 0.
 \end{aligned}$$

$$\begin{aligned}
 b) \quad (2m+1)x - my + (m+1)z &= m-1, \\
 (m-2)x + (m-1)y + (m-2)z &= m, \\
 (2m-1)x + (m-1)y + (2m-1)z &= m.
 \end{aligned}$$

$$\begin{aligned}
 c) \quad (5\lambda+1)x + 2\lambda y + (4\lambda+1)z &= 1+\lambda, \\
 (4\lambda-1)x + (\lambda-1)y + (4\lambda-1)z &= -1, \\
 2(3\lambda+1)x + 2\lambda y + (5\lambda+2)z &= 2-\lambda.
 \end{aligned}$$

$$\begin{aligned}
 435 \quad a) \quad (2c+1)x - cy - (c+1)z &= 2c, \\
 3cx - (2c-1)y - (3c-1)z &= c+1, \\
 (c+2)x - y - 2cz &= 2.
 \end{aligned}$$

$$\begin{aligned}
 b) \quad 2(\lambda+1)x + 3y + \lambda z &= \lambda+4, \\
 (4\lambda-1)x + (\lambda+1)y + (2\lambda-1)z &= 2\lambda+2, \\
 (5\lambda-4)x + (\lambda+1)y + (3\lambda-4)z &= \lambda-1.
 \end{aligned}$$

$$\begin{aligned}
 \text{c) } & dx + (2d-1)y + (d+2)z = 1, \\
 & (d-1)y + (d-3)z = 1+d, \\
 & dx + (3d-2)y + (3d+1)z = 2-d. \\
 \text{d) } & (3a-1)x + 2ay + (3a+1)z = 1, \\
 & 2ax + 2ay + (3a+1)z = a, \\
 & (a+1)x + (a+1)y + 2(a+1)z = a^2.
 \end{aligned}$$

436 Find the equation of the straight line containing the points  $M_1(x_1, y_1)$ ,  $M_2(x_2, y_2)$ .

437 Find a determinantal equation that is necessary and sufficient for the collinearity of the three points  $M_1(x_1, y_1)$ ;  $M_2(x_2, y_2)$ ;  $M_3(x_3, y_3)$ .

438 Find a determinantal equation sufficient for the concurrence of the three lines

$$a_1x + b_1y + c_1 = 0; a_2x + b_2y + c_2 = 0; a_3x + b_3y + c_3 = 0$$

439 Find a determinantal equation sufficient for the concyclicity of the four points

$$M_0(x_0, y_0); M_1(x_1, y_1); M_2(x_2, y_2); M_3(x_3, y_3)$$

(Several points are called concyclic if they all lie on the same circumference.)

- 440 Find the equation of the circle containing the three points  $M_1(2, 1)$ ;  $M_2(1, 2)$ ;  $M_3(0, 1)$ .
- 441 Find the equation of the second degree curve containing the five points  $M_1(0, 0)$ ;  $M_2(1, 0)$ ;  $M_3(-1, 0)$ ;  $M_4(1, 1)$ ;  $M_5(-1, 1)$ .
- 442 Find the equation of the third degree parabola that contains the four points  $M_1(1, 0)$ ;  $M_2(0, -1)$ ;  $M_3(-1, -2)$ ;  $M_4(2, 7)$ .
- 443 Find the equation of the  $n$ -th degree parabola  $y = a_0x^n + a_1x^{n-1} + \dots + a_n$  containing the  $n + 1$  points  $M_0(x_0, y_0)$ ;  $M_1(x_1, y_1)$ ;  $M_2(x_2, y_2)$ ;  $\dots$ ;  $M_n(x_n, y_n)$ .
- 444 Find a determinantal equation sufficient for the coplanarity of the points  $M_1(x_1, y_1, z_1)$ ;  $M_2(x_2, y_2, z_2)$ ;  $M_3(x_3, y_3, z_3)$ ;  $M_4(x_4, y_4, z_4)$ .
- 445 Find the equation of the sphere containing the four points  $M_1(1, 0, 0)$ ;  $M_2(1, 1, 0)$ ;  $M_3(1, 1, 1)$ ;  $M_4(0, 1, 1)$ .
- 446 Under what conditions will the  $n$  points  $M_1(x_1, y_1)$ ;  $M_2(x_2, y_2)$ ;  $M_3(x_3, y_3)$ ;  $\dots$ ;  $M_n(x_n, y_n)$  lie on a straight line?
- 447 Under what conditions will the  $n$  lines  $a_1x + b_1y + c_1 = 0$ ;  $a_2x + b_2y + c_2 = 0$ ;  $\dots$ ;  $a_nx + b_ny + c_n = 0$  be concurrent?



hold. If no such system of relations holds except the system with

$$c_1 = c_2 = \dots = c_m = 0,$$

the solutions are called linearly independent. The solutions themselves can be written as rows of a matrix. Thus the  $m$  solutions (1) can be written in the matrix

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mn} \end{pmatrix} = A.$$

Show that if the matrix  $A$  has rank  $r$ , then the system (1) has  $r$  linearly independent solutions, and all other solutions appearing in the array (1) are linear combinations of these  $r$  solutions.

- 452 Show that if the rank of a system of  $m$  linear homogeneous equations and  $n$  indeterminates is  $r$ , then the system has  $n - r$  linearly independent solutions and all other solutions of this system are linear combinations of these  $n - r$  solutions.

Such a system of  $n - r$  solutions is called a fundamental system of solutions.

453 Decide whether

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 1 & -2 & 3 & -2 & 0 \end{pmatrix}$$

is a fundamental system of solutions of the following system of linear equations:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 0, \\ 3x_1 + 2x_2 + x_3 + x_4 - 3x_5 &= 0, \\ x_2 + 2x_3 + 2x_4 + 6x_5 &= 0, \\ 5x_1 + 4x_2 + 3x_3 + 3x_4 - x_5 &= 0. \end{aligned}$$

454 Find a fundamental system of solutions for the system of equations appearing in 453.

455 Is

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 4 & 0 & 0 & -6 & 2 \end{pmatrix}$$

a fundamental system of solutions of the system in problem 453?

456 Show that if  $A$  is a matrix of rank  $r$  representing in its rows a fundamental system of solutions of a set of linear homogeneous equations, and  $B$  is an arbitrary non-singular  $r$ -th order matrix, then the matrix  $BA$  is also a fundamental system of solutions of the same system of homogeneous equations.

- 458      Let

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{r1} & \alpha_{r2} & \dots & \alpha_{rn} \end{pmatrix}$$

$$\begin{aligned} x_1 &= c_1\alpha_{11} + c_2\alpha_{21} + \dots + c_r\alpha_{r1}, \\ x_2 &= c_1\alpha_{12} + c_2\alpha_{22} + \dots + c_r\alpha_{r2}, \\ &\vdots \\ x_n &= c_1\alpha_{1n} + c_2\alpha_{2n} + \dots + c_r\alpha_{rn} \end{aligned}$$

460 Decide whether  $(11 \ 1 \ -7)$  is a fundamental system of solutions of the system in problem 403 and also find the general solution.



- 461 Find the general solution of the systems in problems 408, 409, 410, 412, 413.
- 462 Use the general solution of the system in problem 453 (notice the answer from problem 459). Note that  $x_1 = -16$ ,  $x_2 = 23$ ,  $x_3 = x_4 = x_5 = 0$  is a particular solution of the system of problem 411. Give the general solution of the latter system.
- 463 Give the general solution of the systems in problems 406, 414, 415.

# CHAPTER IV - PROBLEMS

## MATRICES

### 1. OPERATIONS WITH SQUARE MATRICES

464 Form the products of the following matrices:

$$\begin{aligned} \text{a) } & \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}; & \text{b) } & \begin{pmatrix} 3 & 5 \\ 6 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}; \\ \text{c) } & \begin{pmatrix} 3 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}; & \text{d) } & \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} \cdot \begin{pmatrix} -1 & -2 & -4 \\ -1 & -2 & -4 \\ 1 & 2 & 4 \end{pmatrix}; \\ \text{e) } & \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 & 1 \\ -1 & 1 & 0 \\ 1 & 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{pmatrix}; & \text{f) } & \begin{pmatrix} a & b & c \\ c & b & a \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a & c \\ 1 & b & b \\ 1 & c & a \end{pmatrix}. \end{aligned}$$

465 Carry out the following operations:

$$\begin{aligned} \text{a) } & \begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}^2; & \text{b) } & \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}^3; & \text{c) } & \begin{pmatrix} 3 & 2 \\ -4 & -2 \end{pmatrix}^5; \\ \text{d) } & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n; & \text{e) } & \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}^n. \end{aligned}$$

\*466 If  $\alpha$  is a real number calculate

$$\lim_{n \rightarrow \infty} \begin{pmatrix} 1 & \frac{\alpha}{n} \\ -\frac{\alpha}{n} & 1 \end{pmatrix}^n$$

467 Assuming that  $AB$  are permutable,  $AB = BA$ , establish the following

- a)  $(A + B)^2 = A^2 + 2AB + B^2$ ;
- b)  $A^2 - B^2 = (A + B)(A - B)$ ;
- c)  $(A + B)^n = A^n + \frac{n}{1} A^{n-1}B + \dots + B^n$ .

168 Calculate  $AB - BA$  in each of the cases:

$$\begin{aligned} \text{a) } A &= \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}, & B &= \begin{pmatrix} 4 & 1 & 1 \\ -4 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix}; \\ \text{b) } A &= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix}, & B &= \begin{pmatrix} 3 & 1 & -2 \\ 3 & -2 & 4 \\ -3 & 5 & -1 \end{pmatrix}. \end{aligned}$$

- 469 For each of the matrices  $A$ , find another matrix commutative with  $A$ :

$$\text{a) } A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \quad \text{b) } A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \quad \text{c) } A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 1 & 2 \end{pmatrix}.$$

- 470 In the given cases find  $f(A)$ :

$$\text{a) } f(x) = x^2 - x - 1, \quad A = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix};$$

$$\text{b) } f(x) = x^2 - 5x + 3, \quad A = \begin{pmatrix} 2 & -1 \\ -3 & 3 \end{pmatrix}.$$

- 471 Let  $A$  be an arbitrary second order matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Show that the following relation is satisfied when  $A$  is substituted for  $x$ :

$$x^2 - (a + d)x + (ad - bc) = 0.$$

- 472 Let  $A$  be an arbitrary square matrix. Show that there is a polynomial  $f(x)$  such that the relation  $f(A) = 0$  holds; furthermore that there is a privileged polynomial of this type which divides evenly every other such polynomial.

- \*473 Show that there are no two matrices  $A, B$  such that the relation  $AB - BA = E$  holds.
- 474 Let  $A$  be a matrix such that  $A^k = 0$ . Show that the following relation holds:
- $$(E - A)^{-1} = E + A + A^2 + \dots + A^{k-1}$$
- 475 If the square of a second order matrix is the null matrix, what form must the second order matrix have?
- 476 If the cube of a second order matrix is the null matrix, what form must the second order matrix have?
- 477 If the square of a second order matrix is the identity matrix, what form must the original matrix have?
- 478 Let  $A$  be some second order matrix. Find all second order square matrices  $X$  satisfying the relation  $XA = 0$ .
- 479 Let  $A$  be a given second order matrix. Find all second order square matrices  $X$  satisfying the relation  $X^2 = A$ .
- 480 Find the inverse of each of the following matrices:

$$\text{a) } A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}; \quad \text{b) } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad \text{c) } A = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix};$$

$$d) A = \begin{pmatrix} 1 & 3 & -5 & 7 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad e) A = \begin{pmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix};$$

$$f) A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & \dots & 1 \end{pmatrix}; \quad g) A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 1 & 1 & 3 & 4 \\ 2 & -1 & 2 & 3 \end{pmatrix};$$

$$h) A = \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}; \quad i) A = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \varepsilon & \varepsilon^2 & \dots & \varepsilon^{n-1} \\ 1 & \varepsilon^2 & \varepsilon^4 & \dots & \varepsilon^{2n-2} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \varepsilon^{n-1} & \varepsilon^{2n-2} & \dots & \varepsilon^{(n-1)^2} \end{pmatrix},$$

where  $\varepsilon = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$ ;

$$j) A = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix};$$

$$k) A = \begin{pmatrix} 1 & 3 & 5 & 7 & \dots & 2n-1 \\ 2n-1 & 1 & 3 & 5 & \dots & 2n-3 \\ 2n-3 & 2n-1 & 1 & 3 & \dots & 2n-5 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 3 & 5 & 7 & 9 & \dots & 1 \end{pmatrix};$$

$$l) A = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & c_1 \\ 0 & 1 & 0 & \dots & 0 & c_2 \\ 0 & 0 & 1 & \dots & 0 & c_3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & c_n \\ b_1 & b_2 & b_3 & \dots & b_n & a \end{pmatrix};$$

$$\text{m) } A = \begin{pmatrix} 1 & -x & 0 & \dots & 0 \\ 0 & 1 & -x & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 & -x \\ a_0 & a_1 & a_2 & \dots & a_n \end{pmatrix};$$

$$\text{n) } A = \begin{pmatrix} 1 + \frac{1}{\lambda_1} & 1 & 1 & \dots & 1 \\ 1 & 1 + \frac{1}{\lambda_2} & 1 & \dots & 1 \\ 1 & 1 & 1 + \frac{1}{\lambda_3} & \dots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \dots & 1 + \frac{1}{\lambda_n} \end{pmatrix}.$$

o) If the inverse  $B^{-1}$  is known show how to find the inverse of the bordered matrix

$$\begin{pmatrix} B & U \\ V & a \end{pmatrix}.$$

481 Find the square matrix  $X$  in each of the following matrix equations:

$$\text{a) } \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \cdot X = \begin{pmatrix} 4 & -6 \\ 2 & 1 \end{pmatrix};$$

$$b) X \cdot \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 3 \\ 4 & 3 & 2 \\ 1 & -2 & 5 \end{pmatrix};$$

$$c) \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \cdot X = \begin{pmatrix} 2 & 1 & 0 & \dots & 0 \\ 1 & 2 & 1 & \dots & 0 \\ 0 & 1 & 2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 2 \end{pmatrix};$$

$$d) \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \cdot X \cdot \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ 3 & -1 \end{pmatrix};$$

$$e) \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & -1 & 1 \end{pmatrix} \cdot X \cdot \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 1 & 1 & -1 & \dots & 0 \\ 1 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & \dots & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix};$$

$$f) \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \cdot X = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}; \quad g) X \cdot \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$



482 Show that the relation  $A^{-1}B = BA^{-1}$  follows from the relation  $AB = BA$ .

483 Suppose  $\varphi(x) = \frac{1+x}{1-x}$ ,  $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ .

Calculate  $\varphi(A)$ .

484 Find all square second order matrices with real elements whose cubes are the identity matrix.

485 Find all square second order matrices with real elements whose fourth powers are equal to the identity matrix.

486 Let  $a, b$  be real. Show that the complex numbers under addition and multiplication are isomorphic to the set of matrices of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

487 Show that the set of all second order matrices

$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \quad \text{where } a, b, c, d \text{ have all}$$

possible real values fill out a ring with no zero divisors.

488 Show how to express the product

$(a_1^2 + b_1^2 + c_1^2 + d_1^2)(a_2^2 + b_2^2 + c_2^2 + d_2^2)$  as the sum of four squares of bilinear expressions.

489 Show that each of the following three operations on a matrix can be realized by multiplying the matrix on the left by non-singular matrix:

- a) interchange of two rows;
- b) addition to one row of a multiple of the respective elements of another row;
- c) multiplication of the elements of a row by a non-zero factor.

Corresponding operations on the columns of the matrix can be realized by multiplication on the right by a non-singular matrix.

490 Show that if  $A$  is an arbitrary matrix,  $A$  can be written in the form  $A = PRQ$ , where  $P$  and  $Q$  are non-singular matrices and the matrix  $R$  is diagonal with 1's in the first  $r$  diagonal positions, and 0's elsewhere.

\*491 Show that every matrix of real numbers can be written as the product of matrices of the form  $E + \alpha e_{ik}$ , where  $\alpha$  is a real number,  $E$  is the identity matrix and  $e_{ik}$  is a matrix unit; that is the matrix that has 1 in the  $ik$  position and 0's elsewhere.

\*492 Show that the rank of the product of two square matrices of order  $n$  is no less than  $r_1 + r_2 - n$ , where  $r_1$  and  $r_2$  are the ranks of the factors.

493 Show that every square matrix of rank 1 has the form

$$\begin{pmatrix} \lambda_1 \mu_1 & \lambda_1 \mu_2 & \dots & \lambda_1 \mu_n \\ \lambda_2 \mu_1 & \lambda_2 \mu_2 & \dots & \lambda_2 \mu_n \\ \dots & \dots & \dots & \dots \\ \lambda_n \mu_1 & \lambda_n \mu_2 & \dots & \lambda_n \mu_n \end{pmatrix}.$$

Can all  $\lambda$ 's be 0?

- \*494 Find all third order matrices whose squares are equal to the null matrix.
- \*495 Find all third order matrices whose squares are equal to the identity matrix.
- \*496 Let  $A, B$  be rectangular matrices with the same number of rows. The symbol  $(A, B)$  denotes the matrix obtained by appending the columns of the matrix  $B$  to the columns of the matrix  $A$ . Establish the relation  $\text{rank } (A, B) \leq \text{rank } A + \text{rank } B$ .
- \*497 Let  $A$  be a square  $n$ -th order matrix. Show that if  $A^2 = E$ , then  $\text{rank } (E + A) + \text{rank } (E - A) = n$ .
- \*498 If  $B$  is a diagonal matrix with entries  $+1$  and  $-1$ , and  $A = PBP^{-1}$ , then  $A$  obviously satisfies the relation  $A^2 = E$ . Establish the converse of this assertion, that is, that every matrix satisfying  $A^2 = E$  can be written in the form  $PBP^{-1}$ , where  $P$  is a non-singular matrix.
- 499 Let  $A$  be a square matrix of integers. Under what conditions is the inverse  $A^{-1}$  also an integral matrix?

- 500 Let  $A$  be an arbitrary square non-singular integral matrix. Show that  $A$  can be written in the form  $A = PR$ , where  $P$  is an integral matrix of determinant  $+1$  or  $-1$ ; and  $R$  is an integral triangular matrix; that is a matrix whose subdiagonal elements are all 0, whose diagonal elements are all positive. Moreover, every element above the main diagonal of  $R$  is non-negative and less than the diagonal element on the same row.
- \*501 Let  $k$  be an arbitrary integer. Partition the integral  $n$ -th order square matrices into equivalence classes, so that two elements belong to the same equivalence class if one can be obtained from the other by multiplication by a unimodular integral matrix (of determinant  $+1$  or  $-1$ ). How many equivalence classes are there?
- 502 Let  $A$  be an integral matrix. Show that  $A$  can be written in the form  $A = PRQ$ , where  $P, Q$  are integral unimodular matrices, and  $R$  is an integral diagonal matrix.
- 503 Show that every properly unimodular integral second order square matrix, that is every such matrix of determinant  $+1$ , can be written as a product of positive and negative powers of the fundamental generating matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

- 504 Show that every second order integral unimodular matrix can be written as a product of positive and negative powers of the fundamental generating matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- 505 Let  $A$  be an arbitrary third order integral matrix with positive determinant,  $A \neq E$ ,  $A^2 = E$ . Show that  $A$  can be written in the form  $QCQ^{-1}$ , where  $Q$  is an integral unimodular matrix, and  $C$  has one of the forms

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

## 2. RECTANGULAR MATRICES. CERTAIN INEQUALITIES

- 506 Multiply the two matrices in each part:

$$\begin{aligned} \text{a) } & \begin{pmatrix} 2 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}; \quad \text{b) } \begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}; \\ \text{c) } & \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad (1 \ 2 \ 3); \quad \text{d) } (1 \ 2 \ 3), \quad \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}. \end{aligned}$$

- 507 Multiply the matrix

$$\begin{pmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{pmatrix}$$

by its transpose and find the determinant of the product.

- 508 Multiply the matrix

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$$

by its transpose and apply the theorem on the determinant of a product to obtain an algebraic identity connecting the determinants.

- 509 Show how an  $m$ -th order minor of the product of two matrices is related to the minors of the factors.
- 510 Let  $A$  be an arbitrary matrix of real numbers,  $A'$  be its transpose. Show that every principal minor of the matrix  $A'A$  is non-negative.
- 511 Let  $A$  be a matrix of real numbers, and  $A'$  be its transpose. Show that the rank of  $A'A$ , as well as the rank of  $A$ , is less than  $k$  if every  $k$ -th order principal minor of the matrix  $A'A$  is zero.
- 512 Let  $A$  be an arbitrary matrix,  $A'$  be its transpose. Show that the sum of all principal  $k$ -th order minors of the matrix  $A'A$  is the same as the corresponding sum of the matrix  $AA'$ .

- 513 Choose rectangular matrices properly (see exercise 508); multiply them, and establish the identity

$$(a_1^2 + a_2^2 + \dots + a_n^2)(b_1^2 + b_2^2 + \dots + b_n^2) - (a_1b_1 + a_2b_2 + \dots + a_nb_n)^2 = \sum_{i < k} (a_ib_k - a_kb_i)^2.$$

- 514 Let  $a_i, b_i$  be complex numbers;  $\overline{b_i}$  be the complex conjugate of  $b_i$ . Establish the following identity.

$$\sum_{i=1}^n |a_i|^2 \cdot \sum_{i=1}^n |b_i|^2 - \left| \sum_{i=1}^n a_i \overline{b_i} \right|^2 = \sum_{i < k} |a_ib_k - a_kb_i|^2.$$

- 515 Use the identity of problem 513 to establish the Buniakovsky inequality

$$\left( \sum_{i=1}^n a_i b_i \right)^2 \leq \sum_{i=1}^n a_i^2 \cdot \sum_{i=1}^n b_i^2$$

for arbitrary real  $a_i, b_i$ .

- 516 Establish the corresponding inequality

$$\left| \sum_{i=1}^n a_i \overline{b_i} \right|^2 \leq \sum_{i=1}^n |a_i|^2 \cdot \sum_{i=1}^n |b_i|^2$$

where  $a_i, b_i$  are arbitrary complex numbers.

\*517 Let  $B, C$  be two real rectangular matrices; suppose that  $(B, C) = A$  is a square matrix. (This means that  $B$  is the matrix consisting of the first columns of  $A$ .) Show that  $(\det A)^2 \leq (\det BB')(\det CC')$ .

\*518 If  $A$  is not square the generalization of 517 is as follows:

$$\det A'A \leq (\det B'B) (\det C'C).$$

519 Let  $A$  be the rectangular real matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

$$\text{Show that } \det AA' \leq \sum_{k=1}^n a_{1k}^2 \cdot \sum_{k=1}^n a_{2k}^2 \dots \sum_{k=1}^n a_{mk}^2.$$

520 Let  $A$  be a rectangular matrix of complex numbers,  $A^*$  be the conjugate transpose matrix. Show that the determinant of  $A^*A$  is real, non-negative, and can be zero only if the rank of  $A$  is less than the number of columns it contains.

521 (Generalization of 518.) Suppose  $A = (B, C)$ . Establish the identity

$$\det(A^*A) \leq \det(B^*B) \cdot \det(C^*C).$$



- 522 Let each element of the  $n$ -th order square matrix  $A = [a_{ij}]$  have modulus less than  $M$ :

$$|a_{ik}| \leq M.$$

Establish the identity

$$|\det A| \leq M^n n^{n/2}.$$

- \*523 The following identity is a little harder to establish; it applies only when the elements are real.

$$|\det A| \leq M^n 2^{-n} (n+1)^{\frac{n+1}{2}}.$$

- 524 Show that the result of problem 522 is exact in  
& 525 many cases:  $n = 1, 2, 2^k, 12, \dots$ . Thus the result cannot be improved.

- 526 Show that of all  $n$ -th order matrices with real entries between  $-1$  and  $+1$  inclusive, there is at least one matrix of maximal determinant. Also show that this maximum is divisible by  $2^{n-1}$ .

- \*527 Find the true upper bound for the absolute value of the determinant of a square matrix of real numbers of order 3 or 5, the elements of the matrix being between  $-1$  and  $+1$ .

- \*528 Let  $A = [a_{ij}]$ . The obverse matrix has for  $ik$ -th element the minor of  $A$  obtained by deleting the  $i$ -th row in  $k$ -th column. Show that the obverse of the obverse matrix has for  $ik$ -th element  $a_{ik} \cdot \Delta^{n-2}$ , where  $\Delta = \det A$ .

- \*529 Show that the  $m$ -th order minors of the obverse matrix can be obtained from the corresponding minors of the original matrix by multiplication with the factor  $\Delta^{m-1}$ .
- 530 Show that the obverse of the product of two matrices is equal to the product of the obverses in the same order.
- 531 Let the combinations of  $m$  sets of the integers  $1, 2, \dots, n$  be indexed in some manner, for instance in lexicographic order.

Let  $A = (a_{ik})$  be an  $m$ -th order matrix. Let  $A_{\alpha\beta}$  be an  $m$ -th order minor of  $A$  obtained by taking the rows of the  $\alpha$ -th combination of rows and the columns of the  $\beta$ -th combination of columns. The matrix of all such minors  $(A_{\alpha\beta})$  is called the  $m$ -th compound  $A_{(m)}$ , and has order  $C_m^n$ . In particular  $A_1 = A$ ;  $A_{n-1}$  is the obverse of the matrix  $A$ .

Establish the following identities:

$$(AB)_{(m)} = A_{(m)}B_{(m)}, \quad E_{(m)} = E, \quad (A^{-1})_{(m)} = (A_{(m)})^{-1}.$$

- 532 Let  $A$  be a triangular matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix},$$

If the combinations of  $m$  subsets of the integers  $1, 2, \dots, n$  are suitably numbered, then the matrix  $A_{(m)}$  is also triangular.

- 533 Show that the determinant of the matrix  $A_{(m)}$  is the  $C_{m-1}^{n-1}$  power of the determinant of  $A$ .
- 534 Let the set of all pairs  $(i, k)$ ;  $i = 1, 2, \dots, n$ ;  $k = 1, 2, \dots, m$  be numbered in an arbitrary manner (for example, in lexicographic order). In particular let the number of the pair  $(i_1, k_1)$  be  $\alpha_1$  and let the number of the pair  $(i_2, k_2)$  be  $\alpha_2$ . By the Kronecker product of the matrices  $A, B$  of orders  $n, m$  respectively, we mean the  $nm$ -th order matrix  $C = A \times B$ , with elements

$$c_{\alpha_1 \alpha_2} = a_{i_1 i_2} b_{k_1 k_2}$$

Establish the following relations, where the symbols denote arbitrary matrices of suitable orders:

- a)  $(A_1 \pm A_2) \times B = (A_1 \times B) \pm (A_2 \times B)$ ,
- b)  $A \times (B_1 \pm B_2) = (A \times B_1) \pm (A \times B_2)$ ,
- c)  $(A' \times B') \cdot (A'' \times B'') = (A' \cdot A'') \times (B' \cdot B'')$ .

- \*535 Show that the value of the determinant of the product  $A \times B$  is  $(\det A)^m (\det B)^n$ .
- 536 Suppose the  $mn$ -th order matrices  $A, B$  are partitioned into  $n^2$  boxes in the following way:

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1n} \\ B_{21} & B_{22} & \dots & B_{2n} \\ \dots & \dots & \dots & \dots \\ B_{n1} & B_{n2} & \dots & B_{nn} \end{pmatrix},$$

where  $A_{ik}, B_{ik}$  are all square matrices of order  $m$ . Let  $C = A \cdot B$  be the product of  $A$  and  $B$  and suppose that  $C$  is partitioned conformally so that the boxes are denoted  $C_{ik}$ . Establish the relation

$$C_{ik} = A_{i1}B_{1k} + A_{i2}B_{2k} + \dots + A_{in}B_{nk}.$$

Thus, a proper way to multiply partitioned matrices is to treat the boxes formally as elements of a ring.

- \*537 Let  $C$  be a matrix of order  $mn$ , partitioned into  $n^2$  square boxes each of dimension  $m$ . Let the sub-matrices so obtained be denoted by  $A_{ik}$  and suppose that every two of these sub-matrices are commutative. One can form the "determinant"

$$\sum \pm A_{1x_1} A_{2x_2} \dots A_{nx_n} = B$$

by considering these sub-matrices as elements of a large matrix. This determinant is in fact a matrix of order  $m$ . Show that the determinant of the matrix  $C$  is equal to the determinant of the original matrix  $B$ .

CHAPTER V - PROBLEMS  
POLYNOMIALS AND RATIONAL FUNCTIONS  
OF A SINGLE INDETERMINATE

1. PROPERTIES OF POLYNOMIALS. TAYLOR'S FORMULA.  
MULTIPLE ROOTS

538 Multiply the following products of polynomials:

a)  $(2x^4 - x^3 + x^2 + x + 1)(x^2 - 3x + 1);$

b)  $(x^3 + x^2 - x - 1)(x^2 - 2x - 1).$

539 Find the quotient and remainder for the following problems:

a)  $2x^4 - 3x^3 + 4x^2 - 5x + 6 \div x^2 - 3x + 1;$

b)  $x^3 - 3x^2 - x - 1 \div 3x^2 - 2x + 1.$

540 Find conditions on the coefficients so that  $x^3 + px + q$  be exactly divisible by  $x^2 + mx - 1$ .

541 Find conditions on the coefficients so that the polynomial  $x^4 + px^2 + q$  be exactly divisible by  $x^2 + mx + 1$ .

542 Simplify the following polynomial

$$1 - \frac{x}{1} + \frac{x(x-1)}{1 \cdot 2} - \dots + (-1)^n \frac{x(x-1)\dots(x-n+1)}{n!}.$$

- 543 In the following problems, find the quotient and remainder:

$$\begin{array}{ll} \text{a) } x^4 - 2x^3 + 4x^2 - 6x + 8 & \div x - 1; \\ \text{b) } 2x^5 - 5x^3 - 8x & \div x + 3; \\ \text{c) } 4x^3 + x^2 & \div x + 1 + i; \\ \text{d) } x^3 - x^2 - x & \div x - 1 + 2i. \end{array}$$

- 544 Use synthetic division to calculate  $f(x_0)$ :

$$\begin{array}{ll} \text{a) } f(x) = x^4 - 3x^3 + 6x^2 - 10x + 16, & x_0 = 4; \\ \text{b) } f(x) = x^5 + (1 + 2i)x^4 - (1 + 3i)x^2 + 7, & x_0 = -2 - i. \end{array}$$

- 545 Use synthetic division to expand the polynomial  $f(x)$  in powers of  $x - x_0$ :

$$\begin{array}{ll} \text{a) } f(x) = x^4 + 2x^3 - 3x^2 - 4x + 1, & x_0 = -1; \\ \text{b) } f(x) = x^5 & x_0 = 1; \\ \text{c) } f(x) = x^4 - 8x^3 + 24x^2 - 50x + 90, & x_0 = 2; \\ \text{d) } f(x) = x^4 + 2ix^3 - (1 + i)x^2 - 3x + 7 + i, & x_0 = -i; \\ \text{e) } f(x) = x^4 + (3 - 8i)x^3 - (21 + 18i)x^2 - (33 - 20i)x + 7 + 18i, & x_0 = -1 + 2i. \end{array}$$

- 546 Use synthetic division to expand each of the following into partial fractions:

$$\text{a) } \frac{x^3 - x + 1}{(x - 2)^5}; \quad \text{b) } \frac{x^4 - 2x^2 + 3}{(x + 1)^5}.$$

- \*547 Use synthetic division to write each of the following in powers of  $x$ :
- a)  $f(x + 3)$ , where  $f(x) = x^4 - x^3 + 1$ ;
  - b)  $(x - 2)^4 + 4(x - 2)^3 + 6(x - 2)^2 + 10(x - 2) + 20$ .
- 548 For the given value  $x = x_0$  find the value of the polynomial  $f(x)$  and its derivative  $f'(x)$ .
- a)  $f(x) = x^5 - 4x^3 + 6x^2 - 8x + 10$ ,  $x_0 = 2$ ;
  - b)  $f(x) = x^4 - 3ix^3 - 4x^2 + 5ix - 1$ ;  $x_0 = 1 + 2i$ .
- 549 Find the multiplicity of each number as a root of the polynomial written beside it:
- a) 2 for the polynomial  $x^5 - 5x^4 + 7x^3 - 2x^2 + 4x - 8$ ;
  - b) -2 for the polynomial  $x^5 + 7x^4 + 16x^3 + 8x^2 - 16x - 16$ ?
- 550 For what values of the coefficient  $a$  will the polynomial  $x^5 - ax^2 - ax + 1$  have -1 as a root of multiplicity at least two?
- 551 Find  $a$ ,  $b$  if the trinomial  $ax^4 + bx^3 + 1$  is divisible by  $(x - 1)^2$ .
- 552 Find  $a$ ,  $b$  if the trinomial  $ax^{n+1} + bx^n + 1$  is divisible by  $(x - 1)^2$ .
- \*553 Show that 1 is a three-fold root of each of the following polynomials:
- a)  $x^{2n} - nx^{n+1} + nx^{n-1} - 1$ ;
  - b)  $x^{2n+1} - (2n+1)x^{n+1} + (2n+1)x^n - 1$ ;
  - c)  $(n-2m)x^n - nx^{n-m} + nx^m - (n-2m)$





- 557 Find conditions under which the polynomial  $x^5 + ax^3 + b$  should have a double root different from 0.
- 558 Find conditions that the polynomial  $x^5 + 10a x^3 + 5b x + c$  should have a triple root different from 0.
- 559 Show that the trinomial  $x^n + ax^{n-m} + b$  cannot have a nonzero multiple root of multiplicity 3 or higher.
- 560 Find conditions under which the trinomial  $x^n + ax^{n-m} + b$  should have a nonzero double root.
- \*561 Show that the k-nomial

$$a_1x^{p_1} + a_2x^{p_2} + \dots + a_kx^{p_k}$$

cannot have a nonzero multiple root of multiplicity exceeding  $k - 1$ .

- \*562 Show that every non-zero root of multiplicity  $k - 1$  of the polynomial

$$a_1x^{p_1} + a_2x^{p_2} + \dots + a_kx^{p_k}$$

must satisfy the following conditions

$$a_1x^{p_1}\varphi'(p_1) = a_2x^{p_2}\varphi'(p_2) = \dots = a_kx^{p_k}\varphi'(p_k),$$

where

$$\varphi(t) = (t - p_1)(t - p_2)(t - p_3) \dots (t - p_k),$$

and conversely.

\*563 Show that a polynomial divisible by its own derivative must have the form  $a_0(x - x_0)^n$ .

564 Show that the polynomial

$$1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \dots + \frac{x^n}{n!}$$

cannot have multiple roots.

565 Show that a necessary and sufficient condition that  $x_0$  should be a root of multiplicity  $k$  for the rational function,  $f(x) = \frac{\varphi(x)}{w(x)}$ , it being understood that the denominator  $w(x)$  does not reduce to zero for  $x = x_0$ , is that the following set of relations be satisfied:

$$f(x_0) = f'(x_0) = \dots = f^{(k-1)}(x_0) = 0, \quad f^{(k)}(x_0) \neq 0.$$

566 Show that the rational function  $f(x) = \frac{\varphi(x)}{w(x)}$  can be written in the following form

$$f(x) = f(x_0) + \frac{f'(x_0)}{1} (x - x_0) + \dots + \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n + \frac{F(x)}{w(x)} (x - x_0)^{n+1},$$

where  $f(x)$  is a suitable polynomial. Here it is assumed that  $w(x_0) \neq 0$ . (Taylor's theorem for rational functions.)

- \*567 Show that if  $x_0$  is a root of multiplicity  $k$  of the polynomial  $f_1(x) f_2'(x) - f_2(x) f_1'(x)$ , then  $x_0$  is a root of multiplicity  $k + 1$  of the polynomial  $f_1(x)f_2(x_0) - f_2(x)f_1(x_0)$  provided the latter is not identically zero. The converse is also true.
- \*568 Show that if  $f(x)$  has no multiple roots, then the polynomial  $[f'(x)]^2 - f(x) f''(x)$  cannot have a root of multiplicity greater than  $n - 1$ ,  $n$  being the degree of  $f(x)$ .
- \*569 Construct a particular polynomial  $f(x)$  of degree  $n$  such that the polynomial  $[f'(x)]^2 - f(x) f''(x)$  has a root  $x_0$  of multiplicity of  $n - 1$ , where  $x_0$  is not a root of  $f(x)$ .

## 2. PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA AND RELATED QUESTIONS

- 570 Find a number  $\delta$  with a property that the polynomial  $x^5 - 4x^3 + 2x$  has absolute value less than 0.1 for all values of  $x$  such that  $|x| < \delta$ .
- 571 Let  $f(x) = x^4 - 3x^3 + 4x + 5$ . Find a number  $\delta$  such that the relation  $|f(x) - f(2)| < 0.01$  holds for all values  $x$  that satisfy the inequality  $|x - 2| < \delta$ .
- 572 Find a value of  $M$  so that the relation
- $$|x^4 - 4x^3 + 4x^2 + 2| > 100.$$
- is valid for all values of  $x$  greater than  $M$  in absolute value:  $|x| > M$ .

- 573 Find a variety, or closed set, of values of  $x$  such that the relation  $|f(x)| < |f(0)|$  holds:

a)  $f(x) = x^5 - 3ix^3 + 4$ ;    b)  $f(x) = x^5 - 3x^3 + 4$ .

- 574 Find a large set of values of  $x$  such that the relation  $|f(x)| < |f(1)|$  holds for the following polynomials  $f(x)$ :

a)  $f(x) = x^4 - 4x^3 + 2$ ;  
 b)  $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 5$ ;  
 c)  $f(x) = x^4 - 4x + 5$ .

- 575 Let  $f(z)$  be the polynomial

$$f(z) = (1+i)z^5 + (3-5i)z^4 - (9+5i)z^3 - 7(1-i)z^2 + 2(1+3i)z + 4-i.$$

Supposing that the relation  $z - i = a(1 - i)$ ,

$0 < a < \frac{1}{2}$ , holds, show that  $|f(z)| < \sqrt{5}$ .

- 576 Let  $f(z)$  be an arbitrary non-constant polynomial. Show that in every neighborhood of  $z_0$  there is a number  $z_1$  such that the relation  $|f(z_1)| > |f(z_0)|$  is satisfied.

- 577 Prove D'Alembert's lemma for rational functions.

- 578 Show that the maximum of the absolute value of a rational function of an independent variable is actually obtained when the variable varies over a rectangular region.

- 579 It is clear that the fundamental theorem of algebra (on the existence of roots of a polynomial equation)

is untrue for rational functions. Indeed, the function  $1/z$  is never zero. At what point does the proof of the fundamental theorem algebra fail when we attempt to apply the proof to rational functions?

- \*580 Let  $f(x)$  be a polynomial or rational function. Show that if  $a$  is a  $k$ -th order root of  $f(z) - f(a)$ , and  $f(a) \neq 0$ , then every circumference  $|z - a| = \rho$  contains  $2k$  points satisfying the relation  $|f(z)| = |f(a)|$  provided  $\rho$  is sufficiently small.
- \*581 Let  $f(x)$  be a polynomial or rational function. Show that if  $a$  is a  $k$ -fold root of  $f(z) - f(a)$ , then every sufficiently small circumference  $|z - a| = \rho$  contains  $2k$  points for which  $\operatorname{Re}(f(z)) = \operatorname{Re}(f(a))$ , and  $2k$  points for which  $\operatorname{Im}(f(z)) = \operatorname{Im}(f(a))$ .

3. FACTORING INTO LINEAR FACTORS.  
FACTORING INTO FACTORS THAT ARE IRREDUCIBLE  
IN THE FIELD OF REAL NUMBERS.  
FORMULAS CONNECTING COEFFICIENTS AND ROOTS.

- 582 Factor each of the following polynomials into linear factors.

a)  $x^3 - 6x^2 + 11x - 6$ ; b)  $x^4 + 4$ ; c)  $x^4 + 4x^3 + 4x^2 + 1$ ;  
d)  $x^4 - 10x^2 + 1$ .

\*583 Factor each of the following polynomials into linear factors:

- a)  $\cos(n \arccos x)$ ;
- b)  $(x + \cos \theta + i \sin \theta)^n + (x + \cos \theta - i \sin \theta)^n$ ;
- c)  $x^m - C_2^m x^{m-1} + C_4^m x^{m-2} - \dots + (-1)^m C_{2m}^m$ .

584 Factor each of the following into irreducible factors with real coefficients:

- a)  $x^4 + 4$ ; b)  $x^6 + 27$ ; c)  $x^4 + 4x^3 + 4x^2 + 1$ ;
- d)  $x^{2n} - 2x^n + 2$ ; e)  $x^4 - ax^2 + 1$ ,  $-2 < a < 2$ ;
- f)  $x^{2n} + x^n + 1$ .

585 In each case find a polynomial of lowest possible degree having the roots indicated:

- a) 1 as a double root; 2, 3,  $1 + i$  as simple roots;
- b)  $-1$  as a triple root; 3, 4 as simple roots;
- c)  $i$  as a double root;  $-1 - i$  as a simple root.

(It is not required that the coefficients be real.)

586 Find that the polynomial of lowest possible degree having as roots all roots of unity, of all degrees up to and including  $n$ .

587 Find a polynomial of lowest possible degree with real coefficients having the following roots:

- a) 1 as a double root; 2, 3,  $1 + i$  as simple roots;
- b)  $2 - 3i$  as a triple root;
- c)  $i$  as a double root;  $-1 - i$  as a simple root.



- 593 If  $m, n, p$  are positive integers, show that  
 $x^{3m} + x^{3n+1} + x^{3p+2}$  is divisible by  $x^2 + x + 1$ .
- 594 When will  $x^{3m} + x^{3n+1} + x^{3p+2}$  be divisible by  
 $x^2 - x + 1$ ?
- 595 Under what conditions is  $x^{3m} + x^{3n+1} + x^{3p+2}$   
divisible by  $x^4 + x^2 + 1$ ?
- 596 Under what conditions is  $x^{2m} + x^m + 1$  divisible  
by  $x^2 + x + 1$ ?
- 597 Show that  

$$x^{ka_1} + x^{ka_2+1} + \dots + x^{ka_k+k-1}$$
is divisible by  $x^{k-1} + x^{k-2} + \dots + 1$ .
- 598 For what values of  $m$  will  $(x + 1)^m - x^m - 1$   
be divisible by  $x^2 + x + 1$ ?
- 599 For what values of  $m$  will  $(x + 1)^m + x^m + 1$   
be divisible by  $x^2 + x + 1$ ?
- 600 For what values of  $m$  will  $(x + 1)^m - x^m - 1$   
be divisible by  $(x^2 + x + 1)^2$ ?
- 601 For what values of  $m$  will  $(x + 1)^m + x^m + 1$   
be divisible by  $(x^2 + x + 1)$ ?
- 602 Can the polynomials  $(x + 1)^m + x^m + 1$  and  
 $(x + 1)^m - x^m - 1$  be divisible by  $(x^2 + x + 1)^3$ ?



603 Evaluate the polynomial

$$1 - \frac{x}{1} + \frac{x(x-1)}{1 \cdot 2} - \dots + (-1)^n \frac{x(x-1) \dots (x-n+1)}{1 \cdot 2 \dots n},$$

when  $x$  takes on the sequence of values  $1, 2, \dots, n$ .

(See problem 542.)

604 Let  $X_n$  be the cyclotomic polynomial. For what values of  $m$  is  $X_n(x^m)$  divisible by  $X_n(x)$ ?

Demonstrate the following theorems:

605 If  $f(x^n)$  is divisible by  $x - 1$ , then it is also divisible by  $x^n - 1$ .

606 If  $f(x^n)$  is divisible by  $(x - a)^k$ , then it is also divisible by  $(x^n - a^n)^k$ ,  $a \neq 0$ .

607 If  $F(x) = f_1(x^3) + xf_2(x^3)$  is divisible by  $x^2 + x + 1$ , then  $f_1(x)$ ,  $f_2(x)$  are divisible by  $x - 1$ .

\*608 Let  $f(x)$  be a polynomial with real coefficients such that the relation  $f(x) \geq 0$  holds for all real values of  $x$ . Show that  $f(x) = [\varphi_1(x)]^2 + [\varphi_2(x)]^2$ , where  $\varphi_1(x)$  and  $\varphi_2(x)$  are polynomials with real coefficients.

609 Suppose the polynomial  $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$  has roots  $x_1, \dots, x_n$ . What are the roots of the following polynomials:

a)  $a_0x^n - a_1x^{n-1} + a_2x^{n-2} - \dots + (-1)^na_n;$

b)  $a_nx^n + a_{n-1}x^{n-1} + \dots + a_0;$

$$c) f(a) + \frac{f'(a)}{1}x + \frac{f''(a)}{1 \cdot 2}x^2 + \dots + \frac{f^{(n)}(a)}{n!}x^n;$$

$$d) a_0x^n + a_1bx^{n-1} + a_2b^2x^{n-2} + \dots + a_nb^n?$$

- 610 Find a relation among the coefficients of the cubic equation  $x^3 + px^2 + qx + r = 0$ , if one root is the sum of the other two.
- 611 Check that one root of the equation  $36x^3 - 12x^2 - 5x + 1 = 0$  is equal to the sum of the other two. Solve the equation.
- 612 Find a relation among the coefficients of the fourth degree equation  $x^4 + ax^3 + bx^2 + cx + d = 0$  if the sum of two roots is equal to the sum of the other two.
- 613 Show that an equation that fulfills the condition of problem 612 can be transformed into the form  $y^4 + qy^2 + r = 0$  by the substitution  $x = y + \alpha$ , if  $\alpha$  is properly chosen.
- 614 What relation among the coefficients of the fourth degree equation  $x^4 + ax^3 + bx^2 + cx + d = 0$  guarantees that the product of two roots will be equal to the product of the other two?
- 615 Show that if an equation satisfies the conditions of 614, it can be solved by first dividing by  $x^2$  and then making the substitution  $y = x + \frac{c}{ax}$  ( $a \neq 0$ ).

616 Solve the equations:

a)  $x^4 - 4x^3 + 5x^2 - 2x - 6 = 0$ ;

b)  $x^4 + 2x^3 + 2x^2 + 10x + 25 = 0$ ;

c)  $x^4 + 2x^3 + 3x^2 + 2x - 3 = 0$ ;

d)  $x^4 + x^3 - 10x^2 - 2x + 4 = 0$ ,

by using the tricks in problems 612-615.

617 Find the value of  $\lambda$  such that one of the roots of the equation  $x^3 - 7x + \lambda = 0$  is double another.

618 Find  $a, b, c$  so that they are roots of the equation

$$x^3 - ax^2 + bx - c = 0.$$

619 Find  $a, b, c$  so that they are roots of the equation

$$x^3 + ax^2 + bx + c = 0.$$

620 The sum of two roots of the equation

$$2x^3 - x^2 - 7x + \lambda = 0$$

is 1. Find  $\lambda$ .

621 Let  $x_1, x_2, x_3$  be roots of the equation  $x^3 + px + q = 0$ . Find the relation among the coefficients if the equation  $x_3 = \frac{1}{x_1} + \frac{1}{x_2}$  is valid.

622 Find the sum of the squares of the roots of the polynomial

$$x^n + a_1x^{n-1} + \dots + a_n.$$

- \*623 Knowing the values of  $a_1$ ,  $a_2$  and knowing that the roots of the equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n = 0,$$

are in arithmetic progression, solve the equation.

- 624 Are the roots of the following equations in arithmetic progressions?

a)  $8x^3 - 12x^2 - 2x + 3 = 0;$

b)  $2x^4 + 8x^3 + 7x^2 - 2x - 2 = 0$

- 625 A fourth degree parabola has equation

$$y = x^4 + ax^3 + bx^2 + cx + d.$$

Find a straight line that intercepts this curve in points  $M_1$ ,  $M_2$ ,  $M_3$ ,  $M_4$  so that the three line segments cut off by the curve are equal:  $M_1M_2 = M_2M_3 = M_3M_4$ . Give conditions for the solvability of this problem.

- \*626 Find an equation of the fourth degree with roots:

$$\alpha, \frac{1}{\alpha}, -\alpha, -\frac{1}{\alpha}.$$

- \*627 Find an equation of the sixth degree with roots:

$$\alpha, \frac{1}{\alpha}, 1-\alpha, \frac{1}{1-\alpha}, 1-\frac{1}{\alpha}, \frac{1}{1-\frac{1}{\alpha}}.$$

628 Let  $f(x)$  be the polynomial

$$f(x) = (x - x_1)(x - x_2) \dots (x - x_n).$$

Find formulas for  $f'(x_i)$ ,  $f''(x_i)$  and show that

$$\frac{\partial f'(x_i)}{\partial x_i} = \frac{1}{2} f''(x_i).$$

629 Let  $f(x)$  be the polynomial of problem 628. Suppose that  $f(x_1) = f''(x_1) = 0$ , but  $f'(x_1) \neq 0$ .

Show that

$$\sum_{i=2}^n \frac{1}{x_1 - x_i} = 0.$$

630 Suppose that the roots of the polynomial

$x^n + a_1 x^{n-1} + \dots + a_n$  are in arithmetic progression. Find the value of  $f'(x_1)$ .

#### 4. EUCLIDEAN ALGORITHM

631 Find the greatest common divisor of each pair of polynomials:

- a)  $x^4 + x^3 - 3x^2 - 4x - 1$ ,  $x^3 + x^2 - x - 1$ ;
- b)  $x^5 + x^4 - x^3 - 2x - 1$ ,  $3x^4 + 2x^3 + x^2 + 2x - 2$ ;
- c)  $x^6 - 7x^4 + 8x^3 - 7x + 7$ ,  $3x^5 - 7x^3 + 3x^2 - 7$ ;
- d)  $x^5 - 2x^4 + x^3 + 7x^2 - 12x + 10$ ,  $3x^4 - 6x^3 + 5x^2 + 2x - 2$ ;
- e)  $x^6 + 2x^4 - 4x^3 - 3x^2 + 8x - 5$ ,  $x^5 + x^2 - x + 1$ ;
- f)  $x^5 + 3x^4 - 12x^3 - 52x^2 - 52x - 12$ ,  
 $x^4 + 3x^3 - 6x^2 - 22x - 12$ ;

- g)  $x^5 + x^4 - x^3 - 3x^2 - 3x - 1$  ,  $x^4 - 2x^3 - x^2 - 2x + 1$ ;  
 h)  $x^4 - 10x^2 + 1$  ,  $x^4 - 4\sqrt{2}x^3 + 6x^2 + 4\sqrt{2}x + 1$ ;  
 i)  $x^4 + 7x^3 + 19x^2 + 23x + 10$  ,  $x^4 + 7x^3 + 18x^2 + 22x + 12$ ;  
 j)  $x^4 - 4x^3 + 1$  ,  $x^3 - 3x^2 + 1$ ;  
 k)  $2x^6 - 5x^5 - 14x^4 + 36x^3 + 86x^2 + 12x - 31$  ,  
 $2x^5 - 9x^4 + 2x^3 + 37x^2 + 10x - 14$ ;  
 l)  $3x^6 - x^5 - 9x^4 - 14x^3 - 11x^2 - 3x - 1$  ,  
 $3x^5 + 8x^4 + 9x^3 + 15x^2 + 10x + 9$ .

632 Use the Euclidean algorithm to find polynomials

$M_1(x)$ ,  $M_2(x)$  so as to satisfy the relation

$$f_1(x)M_2(x) + f_2(x)M_1(x) = \delta(x), \quad \text{where}$$

$\delta(x)$  is the greatest common divisor of the polynomials  $f_1(x)$ ,  $f_2(x)$ :

- a)  $f_1(x) = x^4 + 2x^3 - x^2 - 4x - 2$ ,  
 $f_2(x) = x^4 + x^3 - x^2 - 2x - 2$ ;  
 b)  $f_1(x) = x^5 + 3x^4 + x^3 + x^2 + 3x + 1$ ,  
 $f_2(x) = x^4 + 2x^3 + x + 2$ ;  
 c)  $f_1(x) = x^6 - 4x^5 + 11x^4 - 27x^3 + 37x^2 - 35x + 35$ ,  
 $f_2(x) = x^5 - 3x^4 + 7x^3 - 20x^2 + 10x - 25$ ;  
 d)  $f_1(x) = 3x^7 + 6x^6 - 3x^5 + 4x^4 + 14x^3 - 6x^2 - 4x + 4$ ,  
 $f_2(x) = 3x^6 - 3x^4 + 7x^3 - 6x + 2$ ;  
 e)  $f_1(x) = 3x^5 + 5x^4 - 16x^3 - 6x^2 - 5x - 6$ ,  
 $f_2(x) = 3x^4 - 4x^3 - x^2 - x - 2$ ;  
 f)  $f_1(x) = 4x^4 - 2x^3 - 16x^2 + 5x + 9$ ,  
 $f_2(x) = 2x^3 - x^2 - 5x + 4$ .

633 Use the Euclidean algorithm to find polynomials

$M_1(x)$ ,  $M_2(x)$  so as to satisfy the relation

$$f_1(x)M_2(x) + f_2(x)M_1(x) = 1 \quad :$$

- a)  $f_1(x) = 3x^3 - 2x^2 + x + 2$ ,  $f_2(x) = x^2 - x + 1$ ;
- b)  $f_1(x) = x^4 - x^3 - 4x^2 + 4x + 1$ ,  $f_2(x) = x^2 - x - 1$ ;
- c)  $f_1(x) = x^5 - 5x^4 - 2x^3 + 12x^2 - 2x + 12$ ,  
 $f_2(x) = x^3 - 5x^2 - 3x + 17$ ;
- d)  $f_1(x) = 2x^4 + 3x^3 - 3x^2 - 5x + 2$ ,  
 $f_2(x) = 2x^3 + x^2 - x - 1$ ;
- e)  $f_1(x) = 3x^4 - 5x^3 + 4x^2 - 2x + 1$ ,  
 $f_2(x) = 3x^3 - 2x^2 + x - 1$ ;
- f)  $f_1(x) = x^5 + 5x^4 + 9x^3 + 7x^2 + 5x + 3$ ,  
 $f_2(x) = x^4 + 2x^3 + 2x^2 + x + 1$ .

634 Use the method of undetermined coefficients to find polynomials  $M_1(x)$ ,  $M_2(x)$  such that the relation

$$f_1(x)M_2(x) + f_2(x)M_1(x) = 1 \quad \text{holds:}$$

- a)  $f_1(x) = x^4 - 4x^3 + 1$ ,  $f_2(x) = x^3 - 3x^2 + 1$ ;
- b)  $f_1(x) = x^3$ ,  $f_2(x) = (1 - x)^2$ ;
- c)  $f_1(x) = x^4$ ,  $f_2(x) = (1 - x)^4$ .

635 Find polynomials  $M_1(x)$ ,  $M_2(x)$  of lowest possible degree satisfying the following relations:

- a)  $(x^4 - 2x^3 - 4x^2 + 6x + 1)M_1(x) + (x^3 - 5x - 3)M_2(x) = x^4$ ;
- b)  $(x^4 + 2x^3 + x + 1)M_1(x)$   
 $+ (x^4 + x^3 - 2x^2 + 2x - 1)M_2(x) = x^3 - 2x$ .

636 Find a polynomial of lowest possible degree that gives remainder

a)  $2x$  when divided by  $(x - 1)^2$ ;  $3x$  when divided by  $(x - 2)^3$ .

b)  $x^2 + x + 1$  when divided by  $x^4 - 2x^3 + 10x - 7$ ;  
 $2x^2 - 3$  when divided by  $x^4 - 2x^3 - 3x^2 + 13x - 10$ .

\*637 Find polynomials  $M(x)$ ,  $N(x)$  satisfying the relation

$$x^m M(x) + (1 - x)^n N(x) = 1.$$

638 Let  $\delta(x)$  be the greatest common divisor of the polynomials  $f_1(x)$ ,  $f_2(x)$ . Suppose that the relation  $f_1(x)M(x) + f_2(x)N(x) = \delta(x)$  holds. What is the greatest common divisor of  $M(x)$ ,  $N(x)$ ?

639 Find the multiple factors of the following polynomials:

- a)  $x^6 - 6x^4 - 4x^3 + 9x^2 + 12x + 4$ ;
- b)  $x^5 - 10x^3 - 20x^2 - 15x - 4$ ;
- c)  $x^6 - 15x^4 + 8x^3 + 51x^2 - 72x + 27$ ;
- d)  $x^5 - 6x^4 + 16x^3 - 24x^2 + 20x - 8$ ;
- e)  $x^6 - 2x^5 - x^4 - 2x^3 + 5x^2 + 4x + 4$ ;
- f)  $x^7 - 3x^6 + 5x^5 - 7x^4 + 7x^3 - 5x^2 + 3x - 1$ ;
- g)  $x^8 + 2x^7 + 5x^6 + 6x^5 + 8x^4 + 6x^3 + 5x^2 + 2x + 1$ .



## 5. THE INTERPOLATION PROBLEM; RATIONAL FUNCTIONS

640 Use Newton's method to find the polynomial of lowest possible degree that has values given by the tables:

$$\text{a) } \begin{array}{c|cccc} x & 0 & 1 & 2 & 3 & 4 \\ \hline f(x) & 1 & 2 & 3 & 4 & 6 \end{array}; \quad \text{b) } \begin{array}{c|ccccc} x & -1 & 0 & 1 & 2 & 3 \\ \hline f(x) & 6 & 5 & 0 & 3 & 2 \end{array};$$

$$\text{c) } \begin{array}{c|cccc} x & 1 & \frac{9}{4} & 4 & \frac{25}{4} \\ \hline f(x) & 1 & \frac{3}{2} & 2 & \frac{5}{2} \end{array}, \text{ find } f(2); \quad \text{d) } \begin{array}{c|cccc} x & 1 & 2 & 3 & 4 & 6 \\ \hline f(x) & 5 & 6 & 1 & -4 & 10 \end{array}.$$

641 Use Lagrange's formula to find a polynomial with the following table of values:

$$\text{a) } \begin{array}{c|cccc} x & 1 & 2 & 3 & 4 \\ \hline y & 2 & 1 & 4 & 3 \end{array}; \quad \text{b) } \begin{array}{c|cccc} x & 1 & i-1 & -i \\ \hline y & 1 & 2 & 3 & 4 \end{array}.$$

\*642 Find a polynomial  $f(x)$  with the following table of values:

$$\begin{array}{c|cccc} x & 1 & \varepsilon_1 & \varepsilon_2 & \dots & \varepsilon_{n-1} \\ \hline f(x) & 1 & 2 & 3 & \dots & n \end{array}, \quad \varepsilon_k = \cos \frac{2\pi k}{n} + i \sin \frac{2\pi k}{n}.$$

643 A polynomial  $f(x)$  has degree not exceeding  $n - 1$ . The polynomial takes the values  $y_1, y_2, \dots, y_n$  when  $x$  runs through the  $n$ -th roots of 1. Find the value of  $f(0)$ .

- \*644 Establish the following theorem. A necessary and  
sufficient condition that the relation

$$f(x_0) = \frac{1}{n} [f(x_1) + f(x_2) + \dots + f(x_n)]$$

should be valid for every polynomial  $f(x)$  of degree not exceeding  $n - 1$  is that the points  $x_1, x_2, \dots, x_n$  be equally spaced on a circumference of a circle with center  $x_0$ .

- \*645 Let  $\phi(x)$  have roots  $x_1, x_2, \dots, x_n$ . If these are all distinct show that

$$\sum_{i=1}^n \frac{x_i^s}{\varphi'(x_i)} = 0 \quad \text{for } 0 \leq s \leq n-2.$$

- 646 In the preceding problem calculate the sum

$$\sum_{i=1}^n \frac{x_i^{n-1}}{\varphi'(x_i)}$$

- 647 Show that Lagrange's interpolation formula can be  
obtained by using the solution of the following  
system of equations:

$$\begin{aligned} a_0 + a_1 x_1 + \dots + a_{n-1} x_1^{n-1} &= y_1, \\ a_0 + a_1 x_2 + \dots + a_{n-1} x_2^{n-1} &= y_2, \\ &\vdots \\ a_0 + a_1 x_n + \dots + a_{\mu-1} x_n^{n-1} &= y_n. \end{aligned}$$

- \*648 Find the polynomial of lowest possible degree with the following table of values

$$\begin{array}{c|cccc} x & 0 & 1 & 2 & \dots & n \\ \hline y & 1 & 2 & 4 & \dots & 2^n \end{array}.$$

- \*649 Find the polynomial of lowest possible degree with the following table of values

$$\begin{array}{c|cccc} x & 0 & 1 & 2 & \dots & n \\ \hline y & 1 & a & a^2 & \dots & a^n \end{array}.$$

- \*650 Find a polynomial of degree  $2n$  that leaves remainder 1 when divided by  $x(x-2)\cdots(x-2n)$  and leaves remainder  $-1$  when divided by  $(x-1)(x-3)\cdots[x-(2n-1)]$ .
- \*651 Find the polynomial of lowest possible degree with the following table of values

$$\begin{array}{c|cccc} x & 1 & 2 & 3 & \dots & n \\ \hline y & 1 & \frac{1}{2} & \frac{1}{3} & \dots & \frac{1}{n} \end{array}.$$

- \*652 Find a polynomial of degree not exceeding  $(n-1)$  for which the relation  $f(x) = \frac{1}{x-a}$  is valid for each of the values  $x_1, x_2, \dots, x_n$ , ( $x_i \neq a$ ,  $i = 1, 2, \dots, n$ ).
- \*653 Show that if a polynomial has degree  $k \leq n$  and has integral values for  $n+1$  consecutive integral values of the argument, then the polynomial takes integral values for all integral values of its argument.

\*654 Show that if a polynomial has degree  $n$  and takes integral values for  $x = 0, 1, 4, 9, \dots, n^2$ , then the value of the polynomial will be an integer for every integral square value of its argument.

\*655 Resolve each of the following into partial fractions of the first kind:

$$\begin{aligned} \text{a) } & \frac{x^2}{(x-1)(x+2)(x+3)}; & \text{b) } & \frac{1}{(x-1)(x-2)(x-3)(x-4)}; \\ \text{c) } & \frac{3+x}{(x-1)(x^2+1)}; & \text{d) } & \frac{x^2}{x^4-1}; & \text{e) } & \frac{1}{x^3-1}; & \text{f) } & \frac{1}{x^4+4}; \\ \text{g) } & \frac{1}{x^n-1}; & \text{h) } & \frac{1}{x^n+1}; & \text{i) } & \frac{n!}{x(x-1)(x-2)\dots(x-n)}; \\ \text{j) } & \frac{(2n)!}{x(x^2-1)(x^2-4)\dots(x^2-n^2)}; & \text{k) } & \frac{1}{\cos(n \arccos x)}. \end{aligned}$$

\*656 Resolve each of the following into partial fractions of the first or second kind with real coefficients:

$$\begin{aligned} \text{a) } & \frac{1}{x^3-1}; & \text{b) } & \frac{x^2}{x^4-16}; & \text{c) } & \frac{1}{x^4+4}; & \text{d) } & \frac{x^2}{x^6+27}; \\ \text{e) } & \frac{x^m}{x^{2n+1}-1}, & m < 2n+1; & \text{f) } & \frac{x^m}{x^{2n+1}+1}, & m < 2n+1; \\ \text{g) } & \frac{1}{x^{2n}-1}; & \text{h) } & \frac{x^{2m}}{x^{2n}+1}, & m < n; & \text{i) } & \frac{1}{x(x^2+1)(x^2+4)\dots(x^2+n^2)}. \end{aligned}$$

- \*657 Resolve each of the following into partial fractions of the first kind:

$$\begin{aligned} \text{a) } & \frac{x}{(x^2-1)^2}; \quad \text{b) } \frac{1}{(x^2-1)^2}; \quad \text{c) } \frac{5x^2+6x-23}{(x-1)^3(x+1)^3(x-2)}; \quad \text{d) } \frac{1}{(x^n-1)^2}; \\ \text{e) } & \frac{1}{x^m(1-x)^n}; \quad \text{f) } \frac{1}{(x^2-a^2)^n}, \quad a \neq 0; \quad \text{g) } \frac{1}{(x^2+a^2)^n}; \quad \text{h) } \frac{g(x)}{[f(x)]^2}, \end{aligned}$$

where  $f(x) = (x-x_1)(x-x_2)\dots(x-x_n)$  is a polynomial having no multiple roots, and  $g(x)$  is a polynomial of degree not exceeding  $2n$ .

- 658 Resolve each of the following into partial fractions of the first or second kind with real coefficients:

$$\begin{aligned} \text{a) } & \frac{x}{(x+1)(x^2+1)^2}; \quad \text{b) } \frac{2x-1}{x(x+1)^2(x^2+x+1)^2}; \\ \text{c) } & \frac{1}{(x^4-1)^2}; \quad \text{d) } \frac{1}{(x^{2n}-1)^2}. \end{aligned}$$

- 659 Set  $\varphi(x) = (x-x_1)(x-x_2)\dots(x-x_n)$ .

Show how to express each of the following in terms of  $\varphi(x)$ ,  $\varphi'(x)$ ,  $\varphi''(x)$ , etc:

$$\text{a) } \sum \frac{1}{x-x_i}; \quad \text{b) } \sum \frac{x_i}{x-x_i}; \quad \text{c) } \sum \frac{1}{(x-x_i)^2}.$$

- \*660 If  $x_1, x_2, \dots$  are known to be roots of the polynomial  $\varphi(x)$  find the value of the following sums:

a)  $\frac{1}{2-x_1} + \frac{1}{2-x_2} + \frac{1}{2-x_3}, \quad \varphi(x) = x^3 - 3x - 1;$

b)  $\frac{1}{x_1^2 - 3x_1 + 2} + \frac{1}{x_2^2 - 3x_2 + 2} + \frac{1}{x_3^2 - 3x_3 + 2},$   
 $\varphi(x) = x^3 + x^2 - 4x + 1;$

c)  $\frac{1}{x_1^2 - 2x_1 + 1} + \frac{1}{x_2^2 - 2x_2 + 1} + \frac{1}{x_3^2 - 2x_3 + 1}, \quad \varphi(x) = x^3 + x^2 - 1.$

- 661 Find a first degree polynomial that takes values differing as little as possible (in the sense of least squares) from the values in the following table:

$x$	0	1	2	3	4
$y$	2.1	2.5	3.0	3.6	4.1

- 662 Find a second degree polynomial that takes values differing as little as possible (in the sense of least squares) from the values in the following table:

$x$	0	1	2	3	4
$y$	1	1.4	2	2.7	3.6

6. RATIONAL ROOTS OF POLYNOMIALS  
REDUCIBILITY AND IRREDUCIBILITY IN THE FIELD  
OF RATIONAL NUMBERS

- 663 Show that if  $p/q$  is a rational fraction in lowest terms, and is a root of polynomial

$$f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$$

the coefficients of which are integers, then:

- 1)  $q$  divides  $a_0$ ;
- 2)  $p$  divides  $a_n$ ;
- 3)  $p - mq$  divides  $f(m)$  if  $m$  is an integer. In particular  $p - q$  divides  $f(1)$ ;  $p + q$  divides  $f(-1)$ .

- 664 Find the rational roots of the following polynomials:

- a)  $x^3 - 6x^2 + 15x - 14$ ;    b)  $x^4 - 2x^3 - 8x^2 + 13x - 24$ ;
- c)  $x^5 - 7x^3 - 12x^2 + 6x + 36$ ;
- d)  $6x^4 + 19x^3 - 7x^2 - 26x + 12$ ;
- e)  $24x^4 - 42x^3 - 77x^2 + 56x + 60$ ;
- f)  $x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6$ ;
- g)  $24x^5 + 10x^4 - x^3 - 19x^2 - 5x + 6$ ;
- h)  $10x^4 - 13x^3 + 15x^2 - 18x - 24$ ;
- i)  $x^4 + 2x^3 - 13x^2 - 38x - 24$ ;

- \*665 Show that if  $f(x)$  is a polynomial with integral coefficients, and if  $f(0)$ ,  $f(1)$  are both odd, then  $f(x)$  has no integral roots.
- \*666 Let  $x_1, x_2$  be integers. Suppose that a polynomial with rational coefficients takes values  $+1, -1$  when the independent variable is  $x_1, x_2$ . Show that if the relation  $|x_1 - x_2| > 2$  holds, then the polynomial can have no rational root. If this relation does not hold, then the only possible rational root is  $\frac{1}{2}(x_1 + x_2)$ .
- \*667 Use Eisenstein's criterion to show that each of the following polynomials is irreducible:
- a)  $x^4 - 8x^3 + 12x^2 - 6x + 2$ ;  
 b)  $x^5 - 12x^3 + 36x - 12$ ; c)  $x^4 - x^3 + 2x + 1$ .
- \*668 Show that, if  $p$  is a prime, the polynomial
- $$X_p(x) = \frac{x^p - 1}{x - 1}$$
- is irreducible.
- \*669 The same as 668 for the polynomial
- $$X_{p^k}(x) = \frac{x^{p^k} - 1}{x^{p^{k-1}} - 1}$$
- \*670 Let  $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$  be a polynomial with integral coefficients and no rational roots. Show that the polynomial is irreducible if there is a prime  $p$  such that  $a_0$  is not divisible by  $p$ ,  $a_2, a_3, \dots, a_n$  are divisible by  $p$ , and  $a_n$  is not divisible by  $p^2$ .



- \*671 Let  $f(x)$  be a polynomial with integral coefficients; let  $p$  be a prime such that  $a_0$  is not divisible by  $p$ ,  $a_{k+1}, a_{k+2}, \dots, a_n$  are divisible by  $p$ , and  $a_n$  is not divisible by  $p^2$ . Show that if  $f(x)$  has an irreducible divisor, it has one of degree  $n - k$  at least.
- 672 Substitute various integers for  $x$ : factor the polynomial values; in that way factor the corresponding polynomial or show that it is irreducible;
- a)  $x^4 - 3x^2 + 1$ ; b)  $x^4 + 5x^3 - 3x^2 - 5x + 1$ ;  
 c)  $x^4 + 3x^3 - 2x^2 - 2x + 1$ ; d)  $x^4 - x^3 - 3x^2 + 2x + 2$ .
- 673 Show that a third degree polynomial is irreducible if it has no rational root.
- 674 Show that the fourth degree polynomial  $x^4 + ax^3 + bx^2 + cx + d$  with integral coefficients is irreducible if it has no integral roots and is not divisible by a polynomial of the form

$$x^2 + \frac{cm - am^2}{d - m^2} x + m,$$

where  $m$  is a divisor of  $d$ . In applying this criterion one may ignore cases in which the coefficient of  $x$  is not integral. The special polynomials in problems 614-615 form an exception.

- 675 Show that the fifth degree polynomial  $x^5 + ax^4 + bx^3 + cx^2 + dx + e$  with integral coefficients is irreducible if it has no integral roots and has no factor of the form

$$x^2 + \frac{am^3 - cm^2 - dn + be}{m^3 - n^2 + ae - dm} x + m$$

with real coefficients, where  $m$  is a divisor of  $e$ :  $n = \frac{e}{m}$ .

- 676 Use problems 674, 675 to factor each of the following insofar as it is possible:

- a)  $x^4 - 3x^3 + 2x^2 + 3x - 9$ ; b)  $x^4 - 3x^3 + 2x^2 + 2x - 6$ ;  
 c)  $x^4 + 4x^3 - 6x^2 - 23x - 12$ ;  
 d)  $x^5 + x^4 - 4x^3 + 9x^2 - 6x + 6$ .

Note: An application of modular arithmetic to the problem of finding polynomial factors is given in the article: Factorization of the General Polynomial, by D.B. Lloyd. American Math. Monthly 71, (October 1964), 863-870.

See also:

Tables of Irreducible Polynomials for the First Four Prime Moduli, by Randolph Church. Annals of Math. 36 (1935), 198-209.

- 677 Find necessary and sufficient conditions for the reducibility of the polynomial  $x^4 + px^2 + q$  if its coefficients are rational.

- 678 Show that a necessary (but not sufficient) condition that a fourth degree polynomial without rational roots should be reducible is that Ferrari's cubic resolvent should have a rational root.
- 679 Show that if  $a_1, a_2, \dots, a_n$  are distinct integers, then the polynomial

$$f(x) = (x - a_1)(x - a_2) \cdots (x - a_n) - 1$$

is irreducible.

- 680 In problem 679 if the  $-1$  is replaced by  $+1$ , the theorem remains true with the following two exceptions:

$$\begin{aligned} (x-a)(x-a-1)(x-a-2)(x-a-3)+1 &= \\ &= [(x-a-1)(x-a-2)-1]^2 \end{aligned}$$

$$(x-a)(x-a-2)+1 = (x-a-1)^2.$$

- \*681 If an  $n$ -th degree polynomial with integral coefficients takes the values  $+1, -1$  for more than  $2m$  integral values of the argument ( $n = 2m$  or  $2m + 1$ ), then it is irreducible.
- \*682 Show that if the numbers  $a_1, a_2, \dots, a_n$  are distinct prime numbers, the polynomial
- $$f(x) = (x - a_1)^2(x - a_2)^2 \cdots (x - a_n)^2 + 1$$
- is irreducible.
- \*683 Show that if a polynomial  $f(x)$  with integral coefficients assumes the value  $+1$  for more than three different integral values of  $x$ , it cannot assume the value of  $-1$  for any integral value of  $x$ .

- \*684 Show that if  $n \geq 12$  and if an  $n$ -th degree polynomial with integral coefficients takes the values  $+1, -1$  for more than  $n/2$  integral values of its argument, then the polynomial is irreducible.
- \*685 Suppose  $a_1, a_2, \dots, a_n$  are distinct whole numbers,  $n \geq 7$ , and  $\varphi(x) = (x - a_1)(x - a_2) \dots (x - a_n)$ . Show that the polynomial  $ax^2 + bx + 1$  is irreducible,  $a, b$  integral, then the polynomial  $a[\varphi(x)]^2 + b\varphi(x) + 1$  is also irreducible.

# 7. BOUNDS FOR THE ROOTS OF POLYNOMIALS

- 686 Let  $a_0x^n + a_1x^{n-1} + \dots + a_n$  be a polynomial with real or complex coefficients. Show that no root exceeds any of the following quantities in modulus:

$$a) 1 + \max \left| \frac{a_k}{a_0} \right|, \quad k = 1, 2, \dots, n;$$

$$b) \rho + \max \left| \frac{a_k}{a_0 \rho^{k-1}} \right|,$$

$k = 1, 2, \dots, n$ ; and  $\rho$  is an arbitrary positive number;

$$c) 2 \max \sqrt[k]{\left| \frac{a_k}{a_0} \right|}, \quad k = 1, 2, \dots, n;$$

$$d) \left| \frac{a_1}{a_0} \right| + \max \sqrt[k-1]{\left| \frac{a_k}{a_1} \right|}, \quad k = 1, 2, \dots, n.$$

- 687 Show that no root of the polynomial  $a_0x^n + a_1x^{n-1} + \dots + a_n$  can exceed the unique positive root of the polynomial

$b_0x^n - b_1x^{n-1} - b_2x^{n-2} - \dots - b_n$  in modulus,  
provided that  $0 < b_0 \leq |a_0|$ ,  $b_1 \geq |a_1|$ ,  $b_2 \geq |a_2|$ , ...,  $b_n \geq |a_n|$ .

- 688 Show that if  $a_r \neq 0$ , the roots of the polynomial  $f(x) = a_0x^n + a_rx^{n-r} + \dots + a_n$  cannot exceed any of the following quantities in modulus:

$$a) 1 + \sqrt[r]{\max \left| \frac{a_k}{a_0} \right|}, \quad k = r, \dots, n;$$

$$b) \rho + \sqrt[r]{\max \left| \frac{a_k}{a_0 \rho^{k-r}} \right|},$$

$k = r, \dots, n$ ;  $\rho$  is an arbitrary positive number;

$$c) \sqrt[r]{\left| \frac{a_r}{a_0} \right|} + \max \sqrt[k-r]{\left| \frac{a_k}{a_r} \right|}, \quad k = r, \dots, n.$$

- 689 Given any polynomial with real coefficients, construct an auxiliary polynomial according to the following rule. Alter the sign of every coefficient (except the coefficient of the highest power) that has the sign of the coefficient of the highest power. Show that no root of the original polynomial can exceed in modulus the unique non-negative root of the auxiliary polynomial.

Prove the following theorems:

690 Let the polynomial  $a_0x^n + a_1x^{n-1} + \dots + a_n$  have real coefficients,  $a_0 > 0$ . No real root of this polynomial exceeds:

a)  $1 + \sqrt[r]{\max \left| \frac{a_k}{a_0} \right|}$ , where  $r$  is the index of

the first negative coefficient, and the index  $k$  in the max operation ranges over the indices of the negative coefficients.

b)  $\rho + \sqrt[r]{\max \left| \frac{a_k}{a_0 \rho^{k-r}} \right|}$ , where  $r$  is the index of the

first negative coefficient, the index  $k$  ranges over the indices of the negative coefficients, and  $\rho$  is an arbitrary positive number.

c)  $2 \max \sqrt[k]{\left| \frac{a_k}{a_0} \right|}$ , where the index  $k$  ranges

over the indices of the negative coefficients of the polynomial.

d)  $\sqrt[r]{\left| \frac{a_r}{a_0} \right|} + \max \sqrt[k-r]{\left| \frac{a_k}{a_r} \right|}$ , where  $r$  is the index

of the first negative coefficient, and  $k$  ranges over the indices of the negative coefficients.

691 A polynomial with non-negative coefficients cannot have a positive root.

692 If the relations  $f(a) > 0$ ,  $f'(a) \geq 0$ , ...,  $f^{(n)}(a) \geq 0$  hold, every real root of the polynomial  $f(x)$  is less than or equal to  $a$ .

693 Find upper bounds for the real roots of the following polynomials:

- a)  $x^4 - 4x^3 + 7x^2 - 8x + 3$ ; b)  $x^5 + 7x^3 - 3$ ;  
c)  $x^7 - 108x^5 - 445x^3 + 900x^2 + 801$ ;  
d)  $x^4 + 4x^3 - 8x^2 - 10x + 14$ .

## 8. STURM'S THEOREM

694 Construct the Sturm polynomials and isolate the roots of the following polynomials:

- a)  $x^3 - 3x - 1$ ; b)  $x^3 + x^2 - 2x - 1$ ;  
c)  $x^3 - 7x + 7$ ; d)  $x^3 - x + 5$ ; e)  $x^3 + 3x - 5$ .

695 Construct the Sturm polynomials and isolate the roots of the following polynomials:

- a)  $x^4 - 12x^2 - 16x - 4$ ; b)  $x^4 - x - 1$ ;  
c)  $2x^4 - 8x^3 + 8x^2 - 1$ ; d)  $x^4 + x^2 - 1$ ;  
e)  $x^4 + 4x^3 - 12x + 9$ .

- 696 Construct the Sturm polynomials and isolate the roots of the following polynomials:
- a)  $x^4 - 2x^3 - 4x^2 + 5x + 5$ ;    b)  $x^4 - 2x^3 + x^2 - 2x + 1$ ;  
c)  $x^4 - 2x^3 - 3x^2 + 2x + 1$ ;    d)  $x^4 - x^3 + x^2 - x - 1$ ;  
e)  $x^4 - 4x^3 - 4x^2 + 4x + 1$ .
- 697 Construct the Sturm sequence and isolate the roots of the following polynomials:
- a)  $x^4 - 2x^3 - 7x^2 + 8x + 1$ ;    b)  $x^4 - 4x^2 + x + 1$ ;  
c)  $x^4 - x^3 - x^2 - x + 1$ ;    d)  $x^4 - 4x^3 + 8x^2 - 12x + 8$ ;  
e)  $x^4 - x^3 - 2x + 1$ .
- 698 Construct the Sturm sequence and isolate the roots of the following polynomials
- a)  $x^4 - 6x^2 - 4x + 2$ ;    b)  $4x^4 - 12x^2 + 8x - 1$ ;  
c)  $3x^4 + 12x^3 + 9x^2 - 1$ ;    d)  $x^4 - x^3 - 4x^2 + 4x + 1$ ;  
e)  $9x^4 - 126x^2 - 252x - 140$ .
- 699 Construct the Sturm sequence and isolate the roots of the following polynomials:
- a)  $2x^5 - 10x^3 + 10x - 3$ ;  
b)  $x^6 - 3x^5 - 3x^4 + 11x^3 - 3x^2 - 3x + 1$ ;  
c)  $x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$ ;    d)  $x^5 - 5x^3 - 10x^2 + 2$ .



- 700 Construct the Sturm sequence, evaluate the Sturm functions for suitable positive values, and isolate the roots of the polynomials.
- a)  $x^4 + 4x^2 - 1$ ;    b)  $x^4 - 2x^3 + 3x^2 - 9x + 1$ ;  
c)  $x^4 - 2x^3 + 2x^2 - 6x + 1$ ;    d)  $x^5 + 5x^4 + 10x^2 - 5x - 3$ .
- 701 Use Sturm's theorem to determine the number of real roots of the equation  $x^3 + px + q = 0$ , where  $p, q$  are real.
- \*702 Find the number of real roots of the equation  $x^n + px + q = 0$ , where  $p, q$  are real.
- 703 Find the number of real roots of the equation  $x^5 - 5ax^3 + 5a^2x + 2b = 0$ , where  $a, b$  are real.
- 704 Show that if a Sturm sequence contains polynomials of all degrees from 0 to  $n$ , then the number of changes of sign in the sequence of coefficients of the highest powers in the Sturm polynomials is equal to the number of pairs of conjugate complex roots of the original polynomials.
- 705 Suppose that the polynomials  $f(x), f_1(x), f_2(x), \dots, f_k(x)$  satisfy the following hypotheses:
- 1)  $f(x)f_1(x)$  changes sign from  $+$  to  $-$  when  $x$  passes through a root of  $f(x)$ ;
  - 2) two adjacent members of the series of polynomials never vanish simultaneously;

- 3) If  $f_{\lambda}(x_0) = 0$ , then  $f_{\lambda-1}(x_0)$  and  $f_{\lambda+1}(x_0)$  have opposite signs;
- 4) The sequence of polynomials does not change sign in the interval  $(a, b)$ . Show that the number of roots of the polynomial  $f(x)$  in the interval  $(a, b)$  is equal to the increase in the number of changes of sign in the values of the sequence of polynomials  $f_1, \dots, f_k$  as  $x$  changes from  $a$  to  $b$ .

- 706 Let  $x_0$  be a real root of the polynomial  $f'(x)$ ;  
let

$$f_1(x) = \frac{1}{x - x_0} f'(x);$$

let  $f_2(x)$  be the negative of the remainder when  $f(x)$  is divided by  $f_1(x)$ ; let  $f_3(x)$  be the negative of the remainder when  $f_1(x)$  is divided by  $f_2(x)$ ; etc. Suppose that  $f(x)$  has no multiple root. Find the connection between the number of real roots of  $f(x)$  and the number of changes of sign of the sequence of polynomials between  $x = -\infty$ ,  $x = x_0$ ,  $x = +\infty$ .

- \*707 For the Hermite polynomials

$$P_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n e^{-\frac{x^2}{2}}}{dx^n}$$

construct the Sturm sequence and find the number of real roots.

- \*708 For the Laguerre polynomials, find the number of real roots:

$$P_n(x) = (-1)^n e^x \frac{d^n e^{-x} x^n}{dx^n}.$$

Find the number of real roots of the following polynomials:

\*709 
$$E_n(x) = 1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \dots + \frac{x^n}{n!}.$$

\*710 
$$P_n(x) = (-1)^{n+1} x^{2n+2} e^{-\frac{1}{x}} \frac{d^{n+1} \left( e^{\frac{1}{x}} \right)}{dx^{n+1}}.$$

\*711 
$$P_n(x) = \frac{(-1)^n}{n!} (x^2 + 1)^{n+1} \frac{d^n}{dx^n} \left( \frac{1}{x^2 + 1} \right).$$

\*712 
$$P_n(x) = (-1)^n (x^2 + 1)^{n+\frac{1}{2}} \frac{d^n}{dx^n} \left( \frac{1}{\sqrt{x^2 + 1}} \right).$$

- \*713 Let  $f(x)$  be a third degree polynomial without multiple roots. Show that the polynomial  $F(x) = 2f(x) f''(x) - [f'(x)]^2$  has exactly two real roots. Discuss the case when  $f(x)$  has a double or triple root.

- 714 Show that if all roots of the polynomial  $f(x)$  are real and distinct, every root of each of the polynomials in the Sturm sequence constructed by Euclid's algorithm will be real and simple.

# 9. MISCELLANEOUS METHODS FOR THE LOCATION OF ROOTS OF POLYNOMIALS

Prove the following theorems:

- 715 All roots of the Legendre polynomial

$$P_n(x) = \frac{d^n (x^2 - 1)^n}{dx^n}$$

are real and lie in the interval  $(-1, +1)$ .

- 716 If all roots of a polynomial  $f(x)$  with real coefficients are real, then all roots of the polynomial  $\lambda f(x) + f'(x)$  are real, where  $\lambda$  is an arbitrary real number.

- \*717 If all roots of the polynomial  $f(x)$  with real coefficients are real, and all roots of the polynomial  $g(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n$  with real coefficients are real, then all roots of the polynomial

$$F(x) = a_0 f(x) + a_1 f'(x) + \dots + a_n f^{(n)}(x)$$

are also real.

- \*718 If all roots of the polynomial

$$f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n \quad \text{with real}$$

coefficients are real, then all roots of the polynomial

$$a_0 x^n + a_1 m x^{n-1} + a_2 m(m-1) x^{n-2} + \dots + a_n m(m-1) \dots (m-n+1)$$

are real, where  $m$  is an arbitrary positive integer.

- \*719 If all roots of the polynomial

$f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$  with real coefficients are real, then all roots of the polynomial

$$G(x) = a_0x^n + C_1^n a_1x^{n-1} + C_2^n a_2x^{n-2} + \dots + a_n$$

are real.

- 720 Show that all roots of the following polynomial are real

$$x^n + \left(\frac{n}{1}\right)^2 x^{n-1} + \left(\frac{n(n-1)}{1 \cdot 2}\right)^2 x^{n-2} + \dots + 1.$$

- \*721 Find the number of real roots of the following polynomial

$$nx^n - x^{n-1} - x^{n-2} - \dots - 1.$$

- 722 Find the number of real roots of the polynomial

$$x^{2n_1+1} + x^{2n_2+1} + \dots + x^{2n_k+1} + a.$$

- 723 Assuming that  $a, b, c, A, B, C$  are real, find the number of real roots of the polynomial

$$(x-a)(x-b)(x-c) - A^2(x-a) - B^2(x-b) - C^2(x-c).$$

- 724 Suppose that  $a_1, a_2, \dots, a_n, A_1, A_2, \dots, A_n, B$  are real. Show that the polynomial

$$\varphi(x) = \frac{A_1^2}{x-a_1} + \frac{A_2^2}{x-a_2} + \dots + \frac{A_n^2}{x-a_n} + B$$

can have no imaginary roots.

Establish the following theorems:

- 725 If a polynomial  $f(x)$  with real coefficients has roots that are real and distinct, then  $[f'(x)]^2 - f(x)f''(x)$  can have no real root.
- 726 If all the roots and coefficients of the polynomials  $f(x)$ ,  $\varphi(x)$  are real, simple, and interlace (that is, there is a root of the second polynomial between every two roots of the first, and a root of the first polynomial between every two roots of the second), then all roots of the equation  $\lambda f(x) + \mu \varphi(x) = 0$  are real. Here  $\lambda, \mu$  are arbitrary real numbers.
- \*727 (Converse of 726) Suppose that, for every real  $\lambda, \mu$ , all roots of the polynomial  $F(x) = \lambda f(x) + \mu \varphi(x)$  are real. Then the roots of the polynomials  $f(x)$ ,  $\varphi(x)$  interlace.
- \*728 Let  $f(x)$  be a polynomial with real coefficients and no multiple root. If all the roots of  $f'(x)$  are real and distinct, then the number of real roots of polynomial  $[f'(x)]^2 - f(x)f''(x)$  is equal to the number of imaginary roots of the polynomial  $f(x)$ .
- \*729 If  $f_1(x)$ ,  $f_2(x)$  have real coefficients and real roots that interlace, then the roots of their derivatives interlace.
- \*730 If  $f(x)$  is a polynomial with real coefficients and real roots, then all roots of the polynomial

$F(x) = \gamma f(x) + (\lambda + x)f'(x)$  are real if  $\lambda$  is an arbitrary real number and  $\gamma$  satisfies one of the relations  $\gamma > 0$ ,  $\gamma < -n$ .

\*731 Let the coefficients of the polynomials

$$f(x) = a_0 + a_1x + \dots + a_nx^n$$

$$\varphi(x) = b_0 + b_1x + \dots + b_kx^k$$

be real. Let  $f(x)$  have real roots and let  $\varphi(x)$  have real roots all of which lie outside the interval  $(0, n)$ ; then all the roots of the polynomial

$$a_0\varphi(0) + a_1\varphi(1)x + a_2\varphi(2)x^2 + \dots + a_n\varphi(n)x^n$$

are real.

\*732 Let  $\gamma > n-1$  and let all coefficients be real.

If the polynomial  $f(x) = a_0 + a_1x + \dots + a_nx^n$

has real roots, then all roots of the polynomial

$$a_0 + a_1\gamma x + a_2\gamma(\gamma-1)x^2 + \dots + a_n\gamma(\gamma-1)\dots(\gamma-n+1)x^n$$

are real.

\*733 Suppose  $\gamma > n-1$ ,  $\alpha > 0$  and all coefficients

are real. If the roots of the polynomial

$f(x) = a_0 + a_1x + \dots + a_nx^n$  are real, then all

roots of the polynomial

$$a_0 + \frac{\gamma}{\alpha}a_1x + \frac{\gamma(\gamma-1)}{\alpha(\alpha+1)}a_2x^2 + \dots + \frac{\gamma(\gamma-1)\dots(\gamma-n+1)}{\alpha(\alpha+1)\dots(\alpha+n-1)}a_nx^n$$

are real.

- \*734 Let  $0 < \omega \leq 1$ . If all coefficients are real, and the polynomial  $f(x) = a_0 + a_1x + \dots + a_nx^n$  has real roots, then all roots of the polynomial

$$a_0 + a_1\omega x + a_2\omega^4x^2 + \dots + a_n\omega^{n^2}x^n$$

are real.

- \*735 Let the coefficients of the polynomial

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$

be real and suppose its roots all have the same sign. Then all roots of the polynomial

$$a_0 \cos \varphi + a_1 \cos(\varphi + \theta)x + a_2 \cos(\varphi + 2\theta)x^2 + \dots + a_n \cos(\varphi + n\theta)x^n$$

are real. Here  $\varphi, \theta$  are arbitrary real quantities.

- \*736 Let the numbers  $a_0, a_1, \dots, a_n, b_0, b_1, \dots, b_n$  be real. If all roots of the polynomial

$$(a_0 + ib_0) + (a_1 + ib_1)x + \dots + (a_n + ib_n)x^n$$

lie in the upper half plane, then all roots of the polynomials

$$a_0 + a_1x + \dots + a_nx^n, \quad b_0 + b_1x + \dots + b_nx^n$$

are real and interlace.

- \*737 Let  $\varphi(x), \psi(x)$  have real coefficients and suppose all the roots are real and interlace. Then the imaginary parts of the roots of  $\varphi(x) + i\psi(x)$  have constant sign.

- \*738 If all roots of the polynomial  $f(x)$  lie in the upper half plane, then all roots of its derivative lie in the upper half plane.



- \*739 Suppose all roots of a polynomial lie in some half plane. Then the roots of its derivative lie in the same half plane.
- \*740 (Generalization of 739) If the roots of a polynomial lie inside an arbitrary convex contour, then the roots of its derivative lie in the same contour.
- \*741 Let  $f(x)$  be an  $n$ -th degree polynomial with real coefficients and real roots. Then all roots of the polynomial  $[f(x)]^2 + k^2[f'(x)]^2$  have imaginary part not exceeding  $kn$  in absolute value.
- 742 Let the coefficients of a polynomial  $f(x)$  be real; and suppose all roots of the polynomials  $f(x) - a$ ,  $f(x) - b$  are real. Then all roots of the polynomial  $f(x) - \lambda$  are real, where  $a < \lambda < b$ .
- \*743 A necessary and sufficient condition that a polynomial  $x^n + a_1x^{n-1} + \dots + a_n$  with real coefficients should have roots the real parts of which have constant sign is that the roots of the polynomials

$$x^n - a_2x^{n-2} + a_4x^{n-4} - \dots$$

$$a_1x^{n-1} - a_3x^{n-3} + \dots$$

be real and interlace.

- \*744 Let  $a, b, c$  be real. Give necessary and sufficient conditions that the real parts of all roots of the equation  $x^3 + ax^2 + bx + c = 0$  be negative.
- \*745 Let  $a, b, c, d$  be real. Give necessary and sufficient conditions that the real parts of all roots of the equation  $x^4 + ax^3 + bx^2 + cx + d = 0$  be negative.
- \*746 Let  $a, b, c$  be real. Give necessary and sufficient conditions that no root of the equation  $x^3 + ax^2 + bx + c = 0$  exceed 1 in modulus.
- \*747 Show that the conditions  $a_0 \geq a_1 \geq a_2 \geq \dots \geq a_n \geq 0$  are sufficient to guarantee that no root of the polynomial  $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$  exceed 1 in modulus.

#### 10. APPROXIMATE CALCULATION OF THE ROOTS OF POLYNOMIALS

- 748 Calculate to within 0.0001 the root of  $x^3 - 3x^2 - 13x - 7 = 0$  lying in the interval  $(-1, 0)$ .
- 749 Calculate the real root of the polynomial  $x^3 - 2x - 5 = 0$  to within 0.000001.
- 750 Find the real roots of the following polynomials to within 0.0001:
- a)  $x^3 - 10x - 5 = 0$ ; b)  $x^3 + 2x - 30 = 0$ ;  
 c)  $x^3 - 3x^2 - 4x + 1 = 0$ ; d)  $x^3 - 3x^2 - x + 2 = 0$ .

- 751 Divide a hemisphere of radius 1 into two equal parts by a plane parallel to the base.
- 752 Find the positive root of the equation  $x^3 - 5x - 3 = 0$  to within 0.0001.
- 753 Find to within 0.0001 the roots of the equations below lying on the intervals marked:
- a)  $x^4 + 3x^3 - 9x - 9 = 0$ ,  $(1, 2)$ ;
  - b)  $x^4 - 4x^3 + 4x^2 - 4 = 0$ ,  $(-1, 0)$ ;
  - c)  $x^4 + 3x^3 + 4x^2 + x - 3 = 0$ ,  $(0, 1)$ ;
  - d)  $x^4 - 10x^2 - 16x + 5 = 0$ ,  $(0, 1)$ ;
  - e)  $x^4 - x^3 - 9x^2 + 10x - 10 = 0$ ,  $(-4, -3)$ ;
  - f)  $x^4 - 6x^2 + 12x - 8 = 0$ ,  $(1, 2)$ ;
  - g)  $x^4 - 3x^2 + 4x - 3 = 0$ ,  $(-3, -2)$ ;
  - h)  $x^4 - x^3 - 7x^2 - 8x - 6 = 0$ ,  $(3, 4)$ ;
  - i)  $x^4 - 3x^3 + 3x^2 - 2 = 0$ ,  $(1, 2)$ .
- 754 Find to within 0.0001 all real roots of the following equations:

- a)  $x^4 + 3x^3 - 4x - 1 = 0$ ;
- b)  $x^4 + 3x^3 - x^2 - 3x + 1 = 0$ ;
- c)  $x^4 - 6x^3 + 13x^2 - 10x + 1 = 0$ ;
- d)  $x^4 - 8x^3 - 2x^2 + 16x - 3 = 0$ ;
- e)  $x^4 - 5x^3 + 9x^2 - 5x - 1 = 0$ ;
- f)  $x^4 - 2x^3 - 6x^2 + 4x + 4 = 0$ ;
- g)  $x^4 + 2x^3 + 3x^2 + 2x - 2 = 0$ ;
- h)  $x^4 + 4x^3 - 4x^2 - 16x - 8 = 0$ .



# CHAPTER VI - PROBLEMS

## SYMMETRIC FUNCTIONS

### 1. SYMMETRIC FUNCTIONS IN TERMS OF THE ELEMENTARY SYMMETRIC FUNCTIONS.

#### CALCULATION OF THE SYMMETRIC FUNCTIONS OF THE ROOTS OF AN ALGEBRAIC EQUATION.

755 Express each of the following in terms of the elementary symmetric functions:

- a)  $x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3$ ;
- b)  $x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$ ;
- c)  $x_1^4 + x_2^4 + x_3^4 - 2x_1^2x_2^2 - 2x_2^2x_3^2 - 2x_3^2x_1^2$ ;
- d)  $x_1^5x_2^2 + x_1^2x_2^5 + x_1^5x_3^2 + x_1^2x_3^5 + x_2^5x_3^2 + x_2^2x_3^5$ ;
- e)  $(x_1 + x_2)(x_1 + x_3)(x_2 + x_3)$ ;
- f)  $(x_1^2 + x_2^2)(x_1^2 + x_3^2)(x_2^2 + x_3^2)$ ;
- g)  $(2x_1 - x_2 - x_3)(2x_2 - x_1 - x_3)(2x_3 - x_1 - x_2)$ ;
- h)  $(x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2$ .

756 Express each of the following in terms of the elementary symmetric functions:

- a)  $(x_1 + x_2)(x_1 + x_3)(x_1 + x_4)(x_2 + x_3)(x_2 + x_4)(x_3 + x_4)$ ;
- b)  $(x_1x_2 + x_3x_4)(x_1x_3 + x_2x_4)(x_1x_4 + x_2x_3)$ ;
- c)  $(x_1 + x_2 - x_3 - x_4)(x_1 - x_2 + x_3 - x_4)(x_1 - x_2 - x_3 + x_4)$ .

757 Symmetrize each of the following polynomials;  
express the symmetric functions so obtained in  
terms of the elementary symmetric functions:

- |                              |                                    |                                  |
|------------------------------|------------------------------------|----------------------------------|
| a) $x_1^2 + \dots$ ;         | g) $x_1^2 x_2^2 x_3 + \dots$ ;     | n) $x_1^3 x_2 x_3 x_4 + \dots$ ; |
| b) $x_1^3 + \dots$ ;         | h) $x_1^3 x_2 x_3 + \dots$ ;       | o) $x_1^3 x_2^2 x_3 + \dots$ ;   |
| c) $x_1^2 x_2 x_3 + \dots$ ; | i) $x_1^3 x_2^2 + \dots$ ;         | p) $x_1^3 x_2^3 + \dots$ ;       |
| d) $x_1^2 x_2^2 + \dots$ ;   | j) $x_1^4 x_2 + \dots$ ;           | q) $x_1^4 x_2 x_3 + \dots$ ;     |
| e) $x_1^3 x_2 + \dots$ ;     | k) $x_1^5 + \dots$ ;               | r) $x_1^4 x_2^2 + \dots$ ;       |
| f) $x_1^4 + \dots$ ;         | l) $x_1^2 x_2^2 x_3 x_4 + \dots$ ; | s) $x_1^5 x_2 + \dots$ ;         |
|                              | m) $x_1^2 x_2^2 x_3^2 + \dots$ ;   | t) $x_1^6 + \dots$               |

758 Express each of the following in terms of the  
elementary symmetric functions:

- a)  $(-x_1 + x_2 + x_3 + \dots + x_n)^2 + (x_1 - x_2 + x_3 + \dots + x_n)^2 + (x_1 + x_2 - x_3 + \dots + x_n)^2 + \dots + (x_1 + x_2 + x_3 + \dots - x_n)^2$ ;
- b)  $(-x_1 + x_2 + x_3 + \dots + x_n)(x_1 - x_2 + x_3 + \dots + x_n) \dots (x_1 + x_2 + \dots - x_n)$ .

- 759 Express each of the following in terms of the elementary symmetric functions:

$$\begin{array}{ll} \text{a) } \sum_{i > k} (x_i - x_k)^2; & \text{b) } \sum_{i > k} (x_i + x_k)^3; \\ \text{c) } \sum_{i > k} (x_i - x_k)^4; & \text{d) } \sum_{\substack{i > k \\ j \neq i; j \neq k}} (x_i + x_k - x_j)^2. \end{array}$$

- 760 Express the symmetric polynomial with leading term below in terms of the elementary symmetric functions:

$$x_1^2 x_2^2 \dots x_k^2 + \dots$$

- 761 Express the following in terms of the elementary symmetric functions

$$\sum (a_1 x_{i_1} + a_2 x_{i_2} + \dots + a_n x_{i_n})^2.$$

Here the sum is extended over all possible permutations

$i_1, i_2, \dots, i_n$  of the numbers  $1, 2, \dots, n$ .

- 762 Express the following in terms of elementary symmetric functions:

$$\begin{array}{ll} \text{a) } \frac{x_1}{x_2} + \frac{x_2}{x_3} + \frac{x_3}{x_1} + \frac{x_2}{x_1} + \frac{x_3}{x_2} + \frac{x_1}{x_3}; & \text{b) } \frac{(x_1 - x_2)^2}{x_1 + x_2} + \frac{(x_2 - x_3)^2}{x_2 + x_3} + \frac{(x_3 - x_1)^2}{x_3 + x_1}; \\ \text{c) } \left( \frac{x_2}{x_1} + \frac{x_3}{x_2} + \frac{x_1}{x_3} \right) \left( \frac{x_1}{x_2} + \frac{x_2}{x_3} + \frac{x_3}{x_1} \right). \end{array}$$

- 763 Express the following in terms of elementary symmetric functions:

$$\begin{aligned} \text{a) } & \frac{x_1x_2}{x_3x_4} + \frac{x_1x_3}{x_2x_4} + \frac{x_1x_4}{x_2x_3} + \frac{x_2x_3}{x_1x_4} + \frac{x_2x_4}{x_1x_3} + \frac{x_3x_4}{x_1x_2}; \\ \text{b) } & \frac{x_1+x_2}{x_3+x_4} + \frac{x_1+x_3}{x_2+x_4} + \frac{x_1+x_4}{x_2+x_3} + \frac{x_2+x_3}{x_1+x_4} + \frac{x_2+x_4}{x_1+x_3} + \frac{x_3+x_4}{x_1+x_2}. \end{aligned}$$

- 764 Express the following in terms of elementary symmetric functions:

$$\text{a) } \sum \frac{1}{x_i}; \quad \text{b) } \sum \frac{1}{x_i^2}; \quad \text{c) } \sum_{i \neq j} \frac{x_i}{x_j}; \quad \text{d) } \sum_{i \neq j} \frac{x_i^2}{x_j^2}; \quad \text{e) } \sum_{i \neq j} \frac{x_i^2}{x_j}; \quad \text{f) } \sum_{\substack{i \neq j \\ i \neq k \\ j > k}} \frac{x_j x_k}{x_i}.$$

- 765 Calculate the sum of the squares of the roots of the equation

$$x^3 + 2x - 3 = 0.$$

- 766 Calculate  $x_1^3x_2 + x_1x_2^3 + x_2^3x_3 + x_2x_3^3 + x_3^3x_1 + x_3x_1^3$  if the numbers  $x_i$  are the roots of the equation  $x^3 - x^2 - 4x + 1 = 0$ .

- 767 Let the  $x_i$  be roots of the equation

$$x^4 + x^3 - 2x^2 - 3x + 1 = 0.$$

Calculate the value of the symmetric function

$$x_1^3x_2x_3 + \dots$$

- 768 Let  $x_1, x_2, x_3$  be the roots of the equation  $x^3 + px + q = 0$ .

Calculate the following:

$$\begin{aligned} \text{a) } & \frac{x_1}{x_2} + \frac{x_2}{x_3} + \frac{x_3}{x_1} + \frac{x_2}{x_1} + \frac{x_3}{x_2} + \frac{x_1}{x_3}; \\ \text{b) } & x_1^4x_2^2 + x_1^2x_2^4 + x_2^4x_3^2 + x_2^2x_3^4 + x_3^4x_1^2 + x_3^2x_1^4; \end{aligned}$$



- c)  $(x_1^2 - x_2x_3)(x_2^2 - x_1x_3)(x_3^2 - x_2x_1)$ ;  
 d)  $(x_1 + x_2)^4(x_1 + x_3)^4(x_2 + x_3)^4$ ;  
 e)  $\frac{x_1^2}{(x_2+1)(x_3+1)} + \frac{x_2^2}{(x_1+1)(x_3+1)} + \frac{x_3^2}{(x_1+1)(x_2+1)}$ ;  
 f)  $\frac{x_1^2}{(x_1+1)^2} + \frac{x_2^2}{(x_2+1)^2} + \frac{x_3^2}{(x_3+1)^2}$ .

- 769 Find the relation among the coefficients of the cubic equation

$$x^3 + ax^2 + bx + c = 0$$

if the square of one of its roots is equal to the sum of the squares of the other two.

- 770 Establish the following theorem. Necessary and sufficient conditions that all roots of the cubic equation  $x^3 + ax^2 + bx + c = 0$  have negative real part are:

$$a > 0; \quad ab - c > 0; \quad c > 0.$$

- 771 Find the area and radius of the circumscribed circle of the triangle, the lengths of whose sides are the roots of the cubic equation

$$x^3 - ax^2 + bx - c = 0.$$

- \*772 Find a relation among the coefficients of an equation whose roots are equal to the sines of the angles of a triangle.

- 773 Calculate the value of the symmetric function suggested, when  $x_1, \dots$  are roots of the equation  $f(x) = 0$  that is given:

a)  $x_1^4 x_2 + \dots$ ,  $f(x) = 3x^3 - 5x^2 + 1$ ;  
 b)  $x_1^3 x_2^3 + \dots$ ,  $f(x) = 3x^4 - 2x^3 + 2x^2 + x - 1$ ;  
 c)  $(x_1^2 + x_1 x_2 + x_2^2)(x_2^2 + x_2 x_3 + x_3^2)(x_3^2 + x_3 x_1 + x_1^2)$ ,  
 $f(x) = 5x^3 - 6x^2 + 7x - 8$ .

- 774 Express the following symmetric functions in terms of the coefficients of the equation

$$a_0 x^3 + a_1 x^2 + a_2 x + a_3 = 0$$

assuming  $x_1, x_2, x_3$  are its roots:

a)  $a_0^4 (x_1 - x_2)^2 (x_1 - x_3)^2 (x_2 - x_3)^2$ ;  
 b)  $a_0^4 (x_1^2 - x_2 x_3)(x_2^2 - x_1 x_3)(x_3^2 - x_1 x_2)$ ;  
 c)  $\frac{(x_1 - x_2)^2}{x_1 x_2} + \frac{(x_1 - x_3)^2}{x_1 x_3} + \frac{(x_2 - x_3)^2}{x_2 x_3}$ ;  
 d)  $a_0^4 (x_1^2 + x_1 x_2 + x_2^2)(x_2^2 + x_2 x_3 + x_3^2)(x_3^2 + x_3 x_1 + x_1^2)$ .

- 775 Let  $x_1, x_2, \dots, x_n$  be the roots of the polynomial

$$x^n + a_1 x^{n-1} + \dots + a_n.$$

Prove that any symmetric polynomial of  $x_2, x_3, \dots, x_n$  can be expressed as a polynomial in  $x_1$ .

776 Use the result of 775 to solve problems 755 e), 755 g), 774 b), 774 d).

777 Let  $f_k$  be the  $k$ -th elementary symmetric function of  $x_1, x_2, \dots, x_n$ . Calculate

$$\sum_{i=1}^n \frac{\partial f_k}{\partial x_i}$$

778 Suppose one knows how to express the symmetric function  $F(x_1, x_2, \dots, x_n)$  in terms of the elementary symmetric functions. Show how to express

$$\sum_{i=1}^n \frac{\partial F}{\partial x_i}$$

in terms of the elementary symmetric functions.

Establish the following theorems:

779 If  $F(x_1, x_2, \dots, x_n)$  is a symmetric function that satisfies the relation

$$F(x_1 + a, x_2 + a, \dots, x_n + a) = F(x_1, x_2, \dots, x_n),$$

and if  $\Phi(f_1, f_2, \dots, f_n)$  is the formula for expressing  $F$  in terms of the elementary ones, then the relation

$$n \frac{\partial \Phi}{\partial f_1} + (n-1) f_1 \frac{\partial \Phi}{\partial f_2} + \dots + f_{n-1} \frac{\partial \Phi}{\partial f_n} = 0$$

holds, and conversely.

780 Any homogeneous second degree symmetric polynomial that satisfies the first relation in problem 779 can be written in the form

$$\alpha \sum_{i < k} (x_i - x_k)^2$$

for some constant value of  $\alpha$ .

781 Find the general form of a third degree symmetric polynomial that satisfies the first relation in problem 779.

782 Use the result of problem 779 to express

$$\sum_{i < j < k} (x_i - x_j)^2 (x_i - x_k)^2 (x_j - x_k)^2$$

in terms of the elementary symmetric functions.

783 Consider the collection of symmetric polynomials  $F(x_1, x_2, \dots, x_n)$  that satisfy the condition

$$F(x_1, x_2, \dots, x_n) = F(x_1 + a, x_2 + a, \dots, x_n + a)$$

Show that there is a basic set of such polynomials  $\varphi_2, \varphi_3, \dots, \varphi_n$ ,  $n - 1$  in number, such that all remaining polynomials of the set can be written in terms of the polynomials  $\varphi_2, \varphi_3, \dots, \varphi_n$ .

784 (Refer to 783) Show how to express each of the following in terms of  $\varphi_2, \varphi_3$ :

- a)  $(x_1 - x_2)^2 (x_1 - x_3)^2 (x_2 - x_3)^2$ ;
- b)  $(x_1 - x_2)^4 + (x_1 - x_3)^4 + (x_2 - x_3)^4$ .

785 (Refer to 783) Show how to express each of the following in terms of  $\varphi_2, \varphi_3, \varphi_4$ :

- a)  $(x_1 + x_2 - x_3 - x_4)(x_1 - x_2 + x_3 - x_4)(x_1 - x_2 - x_3 + x_4)$ ;
- b)  $(x_1 - x_2)^2 (x_1 - x_3)^2 (x_1 - x_4)^2 (x_2 - x_3)^2 (x_2 - x_4)^2 (x_3 - x_4)^2$ .

## 2. POWER SUMS

786 Use Newton's formula to express  $s_2, s_3, s_4, s_5, s_6$  in terms of the elementary symmetric polynomials.

787 Use Newton's formula to express the elementary symmetric functions  $f_2, f_3, f_4, f_5, f_6$  in terms of the power sums  $s_1, s_2, \dots$ .

788 Find the sum of the fifth powers of the roots of the equation

$$x^6 - 4x^5 + 3x^3 - 4x^2 + x + 1 = 0.$$

789 Find the sum of the eighth powers of the roots of the equation

$$x^4 - x^3 - 1 = 0.$$

790 Find the sum of the tenth powers of the roots of the equation

$$x^3 - 3x + 1 = 0.$$

791 Calculate  $s_1, s_2, \dots, s_n$  for the roots of the equation

$$x^n + \frac{x^{n-1}}{1} + \frac{x^{n-2}}{1 \cdot 2} + \dots + \frac{1}{n!} = 0.$$

792 Establish the formula

$$a^k(x_1^k + x_2^k) = (-1)^k \left[ b^k - \frac{k}{1} b^{k-2} a c + \frac{k(k-3)}{1 \cdot 2} b^{k-4} a^2 c^2 - \frac{k(k-4)(k-5)}{1 \cdot 2 \cdot 3} b^{k-6} a^3 c^3 + \dots \right]$$

where  $x_1, x_2$  are the roots of the quadratic equation  $ax^2 + bx + c = 0$ .

- 793 Let  $x_1, x_2, x_3$  be the roots of an arbitrary third degree polynomial. Establish the following:

$$\frac{s_1^5 - s_5}{s_1^3 - s_3} = \frac{5}{3} (f_1^2 - f_2).$$

- 794 For a fourth degree polynomial, show that if the sum of the roots is 0, then:

$$\frac{s_5}{5} = \frac{s_3}{3} \cdot \frac{s_2}{2}.$$

- 795 For a sixth degree polynomial, show that if  $s_1 = s_3 = 0$ , then

$$\frac{s_7}{7} = \frac{s_5}{5} \cdot \frac{s_2}{2}.$$

- 796 Find an  $n$ -th degree polynomial for which

$$s_1 = s_2 = \dots = s_{n-1} = 0.$$

- 797 Find an  $n$ -th degree polynomial for which

$$s_2 = s_3 = \dots = s_n = 0.$$

- 798 Find an  $n$ -th degree polynomial for which

$$s_2 = 1, s_3 = s_4 = \dots = s_n = s_{n+1} = 0.$$

- 799 Express  $\sum_{i < j} x_i^k x_j^k$  in terms of the power sums.

- \*800 Express  $\sum_{i < j} (x_i + x_j)^k$  in terms of the power sums.

- \*801 Express  $\sum_{i < j} (x_i - x_j)^{2k}$  in terms of the power sums.

802 Show that

$$s_k = \det \begin{bmatrix} f_1 & 1 & & \dots & 0 \\ 2f_2 & f_1 & 1 & \dots & 0 \\ 3f_3 & f_2 & f_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ kf_k & f_{k-1} & f_{k-2} & \dots & f_1 \end{bmatrix}.$$

803 Show that

$$f_k = \frac{1}{k!} \det \begin{bmatrix} s_1 & 1 & 0 & \dots & 0 \\ s_2 & s_1 & 2 & \dots & 0 \\ s_3 & s_2 & s_1 & 3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ s_k & s_{k-1} & s_{k-2} & \dots & s_1 \end{bmatrix}.$$

804 Calculate the value of the determinant

$$\det \begin{bmatrix} x^n & x^{n-1} & x^{n-2} & \dots & 1 \\ s_1 & 1 & 0 & \dots & 0 \\ s_2 & s_1 & 2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ s_n & s_{n-1} & s_{n-2} & \dots & n \end{bmatrix}.$$

- \*805 Calculate  $s_m$  for the roots of the cyclotomic equation  $X_n(x) = 0$ .
- \*806 Show that the only possible values of the functions  $f_2, f_3, f_4$  of the roots of the cyclotomic equation  $X_n(x) = 0$  are 0, +1, -1.
- \*807 Solve the following system of equations:

$$\begin{array}{rcl} x_1 + x_2 + \dots + x_n & = & a, \\ x_1^2 + x_2^2 + \dots + x_n^2 & = & a, \\ \cdot & & \cdot \\ x_1^n + x_2^n + \dots + x_n^n & = & a \end{array}$$

and calculate  $x_1^{n+1} + x_2^{n+1} + \dots + x_n^{n+1}$ .

- ```
*808 Calculate the power sums  $s_1, s_2, \dots, s_n$  for the
      roots of the equation
```

$$x^n + (a+b)x^{n-1} + (a^2+ab+b^2)x^{n-2} + \dots$$

$$\dots + (a^n + a^{n-1}b + \dots + b^n) = 0.$$

- ```
*809 Calculate the power sums  $s_1, s_2, \dots, s_n$  for the
      roots of the equation
```

$$x^n + (a + b) x^{n-1} + (a^2 + b^2) x^{n-2} + \dots + (a^n + b^n) = 0.$$

### 3. TRANSFORMATION OF EQUATIONS

- 810 Let  $x_1, x_2, x_3$  be roots of the equation  
 $x^2 + ax^2 + bx + c = 0$ . Find an equation having  
the following roots:



- a)  $x_1 + x_2, x_2 + x_3, x_3 + x_1$ ;  
 b)  $(x_1 - x_2)^2, (x_2 - x_3)^2, (x_3 - x_1)^2$ ;  
 c)  $x_1^2 - x_2x_3, x_2^2 - x_3x_1, x_3^2 - x_1x_2$ ;  
 d)  $(x_1 - x_2)(x_1 - x_3), (x_2 - x_1)(x_2 - x_3), (x_3 - x_1)(x_3 - x_2)$ ;  
 e)  $x_1^2, x_2^2, x_3^2$ ; f)  $x_1^3, x_2^3, x_3^3$ .

- 811 Let  $\varepsilon = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ , and let  $x_1, x_2, x_3$  be the roots of the equation  $x^3 + ax^2 + bx + c = 0$ . Find the equation that has the following roots

$$(x_1 + x_2\varepsilon + x_3\varepsilon^2)^3, (x_1 + x_2\varepsilon^2 + x_3\varepsilon)^3$$

- 812 Let  $x_1, x_2, x_3$  be the roots of the cubic equation  $x^3 + ax^2 + bx + c = 0$ . Find an equation of lowest possible degree one of whose roots is

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \frac{x_3}{x_1}, \text{ if its coefficients can be}$$

expressed rationally in terms of the coefficients of the given equation.

- 813 Let  $x_1, x_2, x_3$  be the roots of the cubic equation  $x^3 + ax^2 + bx + c = 0$ . Find an equation of lowest possible degree one of whose roots is  $x_1/x_2$ , if

the coefficients of the equation can be expressed rationally in terms of the coefficients of the given equation.

- 814 Let  $x_1, x_2, x_3, x_4$  be roots of the equation  $x^4 + ax^3 + bx^2 + cx + d = 0$ . Find the equation of lowest possible degree so that its coefficients can be expressed rationally in terms of  $a, b, c, d$  and that has the following root:

- a)  $x_1x_2 + x_3x_4$ ;
- b)  $(x_1 + x_2 - x_3 - x_4)^2$ ;
- c)  $x_1x_2$ ;
- d)  $x_1 + x_2$ ;
- e)  $(x_1 - x_2)^2$ .

- 815 Use the answers for 814 a), 814 b) to express the roots of a fourth degree equation in terms of the roots of the resolvent cubic of problem 814 a).

- 816 Find a formula for solving the equation

$$x^4 - 6ax^2 + bx - 3a^2 = 0.$$

- 817 Let  $x_1, x_2, x_3, x_4, x_5$  be roots of the equation  $x^5 + ax + b = 0$ . Find an equation one of whose roots is the following:

$$(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1)(x_1x_3 + x_3x_5 + x_5x_2 + x_2x_4 + x_4x_1)$$

## 4. RESULTANTS AND DISCRIMINANTS

- \*818 (Hermite's rule) Show that the resultant of the polynomials

$$f(x) = x^n + a_1x^{n-1} + \dots + a_n, \quad \varphi(x) = b_0x^m + \dots + b_m$$

is the determinant of the matrix, the rows of which are the coefficients of the remainders obtained when  $\varphi(x), x\varphi(x), \dots, x^{n-1}\varphi(x)$  are divided by  $f(x)$ ; the coefficients of the remainders must be arranged according to ascending powers of  $x$ .

Hint: Note that the remainder  $r_k(x)$  of the division of  $x^{k-1}\varphi(x)$  by  $f(x)$  is equal to the remainder of the division of  $xr_{k-1}(x)$  by  $f(x)$ .

- \*819 (Bézout's rule) Show that the resultant of the polynomials

$$\begin{aligned} \psi_k(x) = & (a_0x^{k-1} + a_1x^{k-2} + \dots + a_{k-1})\varphi(x) - \\ & - (b_0x^{k-1} + b_1x^{k-2} + \dots + b_{k-1})f(x), \quad k = 1, \dots, n \end{aligned}$$

is equal to the determinant of the matrix, the rows of which are the coefficients of the following polynomials of degree  $\leq (n-1)$ :

$$\begin{aligned} f(x) &= a_0x^n + a_1x^{n-1} + \dots + a_n \\ \varphi(x) &= b_0x^n + b_1x^{n-1} + \dots + b_n \end{aligned}$$

Note that

$$\begin{aligned}\psi_1 &= a_0\varphi - b_0f, \\ \psi_k &= x\psi_{k-1} + a_{k-1}\varphi - b_{k-1}f.\end{aligned}$$

\*820 If  $n > m$ , show that the resultant of the polynomials

$$f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n, \quad \varphi(x) = b_0x^m + b_1x^{m-1} + \dots + b_m$$

is the determinant of the matrix whose rows are the coefficients of the polynomials  $\chi_k(x)$  of degree  $k \leq (n - 1)$  defined by the following formulas:

$$\begin{aligned}\chi_k(x) &= x^{k-1}\varphi(x) \quad \text{for } 1 \leq k \leq n - m; \\ \chi_k(x) &= (a_0x^{k-n+m-1} + a_1x^{k-n+m-2} + \dots + a_{k-n+m-1})x^{n-m}\varphi(x) - \\ &\quad - (b_0x^{k-n+m-1} + b_1x^{k-n+m-2} + \dots + b_{k-n+m-1})f(x)\end{aligned}$$

It is understood that the polynomials  $\chi_k$  are written in terms of ascending powers of  $x$ .

Note that

$$\begin{aligned}\chi_{n-m+1} &= a_0x^{n-m}\varphi(x) - b_0f(x), \\ \chi_k &= x\chi_{k-1} + a_{k-n+m-1}x^{n-m}\varphi(x) - b_{k-n+m-1}f(x)\end{aligned}$$

for  $k > n - m + 1$ .

821 Find the resultants of the following pairs of polynomials:

- a)  $x^3 - 3x^2 + 2x + 1$  ,  $2x^2 - x - 1$ ;
- b)  $2x^3 - 3x^2 + 2x + 1$  ,  $x^2 + x + 3$ ;
- c)  $2x^3 - 3x^2 - x + 2$  ,  $x^4 - 2x^2 - 3x + 4$ ;
- d)  $3x^3 + 2x^2 + x + 1$  ,  $2x^3 + x^2 - x - 1$ ;
- e)  $2x^4 - x^3 + 3$  ,  $3x^3 - x^2 + 4$ ;
- f)  $a_0x^2 + a_1x + a_2$  ,  $b_0x^2 + b_1x + b_2$ .

822 For what values of  $\lambda$  will the following pairs of polynomials have a common root:

- a)  $x^3 - \lambda x + 2$  ,  $x^2 + \lambda x + 2$ ;
- b)  $x^3 - 2\lambda x + \lambda^3$  ,  $x^2 + \lambda^2 - 2$ ;
- c)  $x^3 + \lambda x^2 - 9$  ,  $x^3 + \lambda x - 3$ ?

823 Eliminate  $x$  from the following system of two equations:

- a)  $x^2 - xy + y^2 = 3$ ,  $x^2y + xy^2 = 6$ ;
- b)  $x^3 - xy - y^3 + y = 0$ ,  $x^2 + x - y^2 - 1 = 0$ ;
- c)  $y = x^3 - 2x^2 - 6x + 8$ ,  $y = 2x^3 - 8x^2 + 5x + 2$ .

824 Solve each of the following five systems:

- a)  $y^2 - 7xy + 4x^2 + 13x - 2y - 3 = 0$ ,  
 $y^2 - 14xy + 9x^2 + 28x - 4y - 5 = 0$ ;
- b)  $y^2 + x^2 - y - 3x = 0$ ,  
 $y^2 - 6xy - x^2 + 11y + 7x - 12 = 0$ ;
- c)  $5y^2 - 6xy + 5x^2 - 16 = 0$ ,  
 $y^2 - xy + 2x^2 - y - x - 4 = 0$ ;
- d)  $y^2 + (x - 4)y + x^2 - 2x + 3 = 0$ ,  
 $y^3 - 5y^2 + (x + 7)y + x^3 - x^2 - 5x - 3 = 0$ ;
- e)  $2y^3 - 4xy^2 - (2x^2 - 12x + 8)y + x^3 + 6x^2 - 16x = 0$ ,  
 $4y^3 - (3x + 10)y^2 - (4x^2 - 24x + 16)y - 3x^3 +$   
 $+ 2x^2 - 12x + 40 = 0$ .

825 Obtain the resultant of the following polynomials

$$a_0x^n + a_1x^{n-1} + \dots + a_n, \quad a_0x^{n-1} + a_1x^{n-2} + \dots + a_{n-1}.$$

826 Establish the relation

$$\Re(f, \varphi_1 \cdot \varphi_2) = \Re(f, \varphi_1) \cdot \Re(f, \varphi_2).$$

\*827 Find the resultant of the polynomials

$$X_n, \quad x^m - 1.$$

\*828 Find the resultant of the polynomials  $X_m, X_n$   
 (cyclotomic polynomials).

829 Calculate the discriminant of each of the five polynomials below:

- a)  $x^3 - x^2 - 2x + 1$ ;      b)  $x^3 + 2x^2 + 4x + 1$ ;  
 c)  $3x^3 + 3x^2 + 5x + 2$ ;      d)  $x^4 - x^3 - 3x^2 + x + 1$ ;  
 e)  $2x^4 - x^3 - 4x^2 + x + 1$ .

830 Calculate the discriminant of each of the polynomials below:

- a)  $x^5 - 5ax^3 + 5a^2x - b$ ;   b)  $(x^2 - x + 1)^3 - \lambda(x^2 - x)^2$ ;  
c)  $ax^3 - bx^2 + (b - 3a)x + a$ ;  
d)  $x^4 - \lambda x^3 + 3(\lambda - 4)x^2 - 2(\lambda - 8)x - 4$ .

831 For what values of  $\lambda$  will the following polynomials have multiple roots:

- a)  $x^3 - 3x + \lambda$ ;   b)  $x^4 - 4x + \lambda$ ;  
c)  $x^3 - 8x^2 + (13 - \lambda)x - (6 + 2\lambda)$ ;  
d)  $x^4 - 4x^3 + (2 - \lambda)x^2 + 2x - 2$ ?

832 Show how the number of real roots of the polynomial with real coefficients is characterized by the sign of its discriminant in the following cases:

- a) a third-degree polynomial;  
b) a fourth-degree polynomial;  
c) a general polynomial.

833 Calculate the discriminant of the polynomial  $x^n + a$ .

\*834 Calculate the discriminant of the polynomial  $x^n + px + q$ .

- \*835 Calculate the discriminant of the polynomial

$$a_0x^{m+n} + a_1x^m + a_2.$$

- 836 Knowing the discriminant of the polynomial

$$a_0x^n + a_1x^{n-1} + \dots + a_n,$$

find the discriminant of the polynomial

$$a_nx^n + a_{n-1}x^{n-1} + \dots + a_0.$$

- 837 Show that the discriminant of a fourth-degree polynomial is equal to the discriminant of its Ferrari resolvent [see problems 814 a) and 80].

- 838 If  $D$  is the discriminant, show that

$$D((x-a)f(x)) = D(f(x))[f(a)]^2.$$

- \*839 Calculate the discriminant of the polynomial

$$x^{n-1} + x^{n-2} + \dots + 1.$$

- \*840 Calculate the discriminant of the polynomial

$$x^n + ax^{n-1} + ax^{n-2} + \dots + a.$$

- 841 Show that the discriminant of the product of two polynomials is equal to the product of the discriminants diminished by the square of their resolvent polynomial.

- 842 Find the discriminant of the polynomial

$$X_{p^m} = \frac{x^{p^m} - 1}{x^{p^{m-1}} - 1}.$$



- \*843 Find the discriminant of the cyclotomic polynomial  $X_n$ .

- \*844 Calculate the discriminant of the polynomial

$$E_n = n! \left( 1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \dots + \frac{x^n}{n!} \right).$$

- \*845 Calculate the discriminant of the polynomial

$$F_n = x^n + \frac{a}{1} x^{n-1} + \frac{a(a-1)}{1 \cdot 2} x^{n-2} + \dots + \frac{a(a-1) \dots (a-n+1)}{n!}.$$

- \*846 Calculate the discriminant of the Hermite polynomial

$$P_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n e^{-\frac{x^2}{2}}}{dx^n}.$$

- \*847 Calculate the discriminant of the Laguerre polynomial

$$P_n(x) = (-1)^n \frac{d^n (x^n e^{-x})}{dx^n}.$$

- \*848 Calculate the discriminant of the Čebyšev polynomial

$$2 \cos \left( n \arccos \frac{x}{2} \right).$$

- \*849 Calculate the discriminant of the polynomial

$$P_n(x) = \frac{(-1)^n}{n!} (1+x^2)^{n+1} \frac{d^n \left( \frac{1}{1+x^2} \right)}{dx^n}.$$

- \*850 Calculate the discriminant of the polynomial

$$P_n(x) = (-1)^n (1+x^2)^{n+\frac{1}{2}} \frac{d^n}{dx^n} \frac{1}{\sqrt{1+x^2}}.$$

- \*851 Calculate the discriminant of the polynomial

$$P_n(x) = (-1)^n x^{2n+2} e^{-\frac{1}{x}} \frac{d^n}{dx^n} \left( e^{\frac{1}{x}} \right).$$

- \*852 Calculate the maximum possible value of the discriminant of the polynomial

$$x^n + a_1 x^{n-1} + \dots + a_n,$$

if all its roots are real and satisfy the relation

$$x_1^2 + x_2^2 + \dots + x_n^2 = n(n-1)R^2.$$

- 853 Find the discriminant of  $f(x^2)$  assuming the discriminant of  $f(x)$  is known.

- 854 Find the discriminant of  $f(x^m)$  assuming the discriminant of  $f(x)$  is known.

- 855 Let  $\varphi(x)$  be a polynomial of degree  $m$ ; let  $x_1, x_2, \dots, x_n$  be the roots of the polynomial  $f(x)$ . Assume that the coefficient of the highest power of  $x$  in both  $f$  and  $\varphi$  is unity.

Show that the discriminant of  $F(x) = f(\varphi(x))$  is given by the formula

$$[D(f)]^m \prod_{i=1}^n D(\varphi(x) - x_i)$$

## 5. TSCHIRNHAUS TRANSFORMATION AND THE ELIMINATION OF IRRATIONALITIES IN THE DENOMINATOR.

856 Use the substitution  $y = x^2 - x - 1$  to transform the equation  $(x - 1)(x - 3)(x + 4) = 0$ .

857 Use the given substitutions to transform the given equations:

a) $x^3 - 3x - 4 = 0$	$y = x^2 + x + 1;$
b) $x^3 + 2x^2 + 2 = 0$	$y = x^2 + 1;$
c) $x^4 - x - 2 = 0$	$y = x^3 - 2;$
d) $x^4 - x^3 - x^2 + 1 = 0$	$y = x^3 + x^2 + x + 1.$

858 Make the given Tschirnhaus transformation in the given equations, and find the inverse transformation:

a) $x^3 - x + 2 = 0,$	$y = x^2 + x;$
b) $x^4 - 3x + 1 = 0,$	$y = x^3 + x;$
c) $x^4 + 5x^3 + 6x^2 - 1 = 0,$	$y = x^3 + 4x^2 + 3x - 1.$

859 Use the substitution  $y = 2 - x^2$  to transform the equation  $x^3 - x^2 - 2x + 1 = 0$  and give an interpretation of the final result.

860 Show that a necessary and sufficient condition that each root of a cubic equation with rational coefficients should be expressible rationally (with rational coefficients) in terms of any other

roots is that the discriminant should be the square of a rational number.

861 Express with no irrationalities in the denominator:

$$\text{a) } \frac{1}{1 + \sqrt{2} - \sqrt{3}}; \quad \text{b) } \frac{1}{1 + \sqrt[3]{2} + 2\sqrt[3]{4}}; \quad \text{c) } \frac{7}{1 - \sqrt[4]{2} + \sqrt{2}}.$$

862 If  $\alpha$  satisfies the equation given write the corresponding quantity with rational denominator:

$$\begin{aligned} \text{a) } & \frac{\alpha}{\alpha + 1}, & \alpha^3 - 3\alpha + 1 &= 0; \\ \text{b) } & \frac{\alpha^2 - 3\alpha - 1}{\alpha^2 + 2\alpha + 1}, & \alpha^3 + \alpha^2 + 3\alpha + 4 &= 0; \\ \text{c) } & \frac{1}{3\alpha^3 + \alpha^2 - 2\alpha - 1}, & \alpha^4 - \alpha^3 + 2\alpha + 1 &= 0, \\ \text{d) } & \frac{1}{\alpha^3 + 3\alpha^2 + 3\alpha + 2}, & \alpha^4 + \alpha^3 - 4\alpha^2 - 3\alpha + 2 &= 0. \end{aligned}$$

863 Let  $x_1$  be a root of the cubic equation  $x^3 + ax^2 + bx + c = 0$ . Show that every rational function of  $x_1$  can be written in the form

$$\frac{Ax_1 + B}{Cx_1 + D}, \quad \text{where } A, B, C, D \text{ are expressible}$$

rationally in terms of:  $x_1$ ,  $a$ ,  $b$ ,  $c$  and the coefficients of the original rational function.

- 864 Let a cubic equation have rational coefficients, and be irreducible over the field of rational numbers; suppose its discriminant is rational. What sort of relation must subsist among the coefficients  $\alpha, \beta, \gamma, \delta$ , if the roots  $x_1, x_2$  are connected by the rational relation

$$y_1, y_2, x_2 = \frac{\alpha x_1 + \beta}{\gamma x_1 + \delta}.$$

- 865 Carry out the transformation  $y = x^2$  in the equation

$$a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0.$$

- 866 Carry out the transformation  $y = x^3$  in the equation

$$a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0.$$

- \*867 Let the polynomial

$$f(x) = x^n + a_1 x^{n-1} + \dots + a_n, \quad a_n \neq 0$$

have integral coefficients and suppose all its

roots  $x_i$  satisfy the condition  $|x_i| \leq 1$ .

Show that it is necessarily true that all roots are roots of unity.

6. POLYNOMIALS THAT ARE INVARIANT UNDER EVEN PERMUTATION OF THE VARIABLES. POLYNOMIALS THAT ARE INVARIANT UNDER CIRCULAR PERMUTATION OF THE VARIABLES.

- 868 Let a polynomial be invariant under even permutations of the variables and change sign under odd

permutations of the variables. Show that the polynomial is divisible by the Vandermondian (Vandermonde determinant in the independent variables) and the quotient of the division is a symmetric polynomial.

- 869 Show that if a polynomial does not change under even permutation of the variables, then it can be written in the form

$$F_1 + F_2 \Delta ,$$

where  $F_1, F_2$  are symmetric polynomials and  $\Delta$  is the Vandermondian.

- 870 Calculate

$$\det \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-2} & x_1^{n+1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-2} & x_2^{n+1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-2} & x_n^{n+1} \end{bmatrix}.$$

- 871 Let  $x_1, x_2, x_3$  be roots of the equation  $x^3 + ax^2 + bx + c = 0$ . Find an equation having

$\alpha x_1 + \beta x_2 + \gamma x_3, \alpha x_2 + \beta x_3 + \gamma x_1, \alpha x_3 + \beta x_1 + \gamma x_2$  as roots.

- 872 Let  $x_1, x_2, x_3$  be roots of the equation  $x^3 + px + q = 0$ ;  $y_1, y_2, y_3$  roots of the equation  $y^3 + p'y + q' = 0$ . Find an equation having the quantities  $x_1y_1 + x_2y_2 + x_3y_3, x_1y_2 + x_2y_3 + x_3y_1, x_1y_3 + x_2y_1 + x_3y_2$  as roots.

- 873 Prove that the following complicated condition is a necessary and sufficient one that the equations

$$\begin{aligned}x^3 + px + q &= 0, \\ y^3 + p'y + q' &= 0\end{aligned}$$

with discriminants  $\Delta$ ,  $\Delta'$  should be linked by a Tschirnhaus transformation. One of the numbers  $\Delta$ ,  $\Delta'$  is the square of a rational number, and one of the equations

$$u^3 = 3pp'u + \frac{27qq' \pm \sqrt{\Delta\Delta'}}{2}$$

should have rational roots.

- 874 Let a polynomial in the  $n$  indeterminates  $x_1, x_2, \dots, x_n$  be invariant under circular permutations of the variables. Show that the polynomial can be expressed in the form

$$\sum A f_1^{\alpha_0} \eta_1^{\alpha_1} \eta_2^{\alpha_2} \dots \eta_{n-1}^{\alpha_{n-1}},$$

where  $\eta_1, \eta_2, \dots, \eta_{n-1}$  are the linear forms:

$$\begin{aligned}\eta_1 &= x_1\varepsilon + x_2\varepsilon^2 + \dots + x_n, \\ \eta_2 &= x_1\varepsilon^2 + x_2\varepsilon^4 + \dots + x_n, \\ &\dots \dots \dots \dots \dots \dots \dots \\ \eta_{n-1} &= x_1\varepsilon^{n-1} + x_2\varepsilon^{2n-2} + \dots + x_n; \quad \varepsilon = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n},\end{aligned}$$

and where the individual terms in the sums satisfy the condition that the sum  $\alpha_1 + 2\alpha_2 + \dots + (n-1)\alpha_{n-1}$  is divisible by  $n$ .

- 875 Find a set of  $n$  fundamental rational functions (fractions with rational coefficients) in terms of which all rational functions that are invariant under circular permutations of the variables can be rationally expressed.
- 876 Show that if the number of variables is three, the fundamental set of problem 875 can be taken as a set of three functions with rational coefficients.
- 877 When the number of variables is four, find a set of four fundamental functions with rational coefficients, as in problem 876.
- 878 When the number of variables is five, find a set of five fundamental functions with rational coefficients, as in problem 877.



## CHAPTER VII - PROBLEMS

### LINEAR ALGEBRA

We use the following notation in this chapter. A space means a vector space over the field of rational numbers unless otherwise specified. The space may be one originally defined or may be a part of a more general space. In the latter case it is usually called a subspace. A linear manifold is the collection of all vectors  $X_0 + X$ , where  $X_0$  is some fixed vector and  $X$  runs through the set of all vectors lying in some subspace.

The notation  $X = (x_1, x_2, \dots, x_n)$  means that the vector  $X$  has the numbers  $x_1, x_2, \dots, x_n$  as coordinates, when some fixed set of vectors is used as the basis for the space in question. When the space is the usual Euclidean one, the basis will be taken to be orthonormal.

A vector is called a point; a one-dimensional manifold a line; a two-dimensional manifold a plane.

#### 1. SUBSPACES AND LINEAR VARIETIES. COORDINATE SUBSPACES.

879 A vector space is spanned by the vectors given in each of the three problems below. Find a basis, and the dimension of the space.

a)  $X_1 = (2, 1, 3, 1), X_2 = (1, 2, 0, 1), X_3 = (-1, 1, -3, 0);$

$$\begin{array}{ll} \text{b) } X_1 = (2, 0, 1, 3, -1) & X_2 = (1, 1, 0, -1, 1), \\ X_3 = (0, -2, 1, 5, -3), & X_4 = (1, -3, 2, 9, -5); \end{array}$$

$$\begin{array}{ll} \text{c) } X_1 = (2, 1, 3, -1), & X_2 = (-1, 1, -3, 1), \\ X_3 = (4, 5, 3, -1), & X_4 = (1, 5, -3, 1). \end{array}$$

880 In each of the three problems two spaces are given spanned respectively by the vectors  $X$  and the vectors  $Y$ . Form the sum and intersection of the spaces; find the dimensions of each; give a basis for each.

$$\begin{array}{ll} \text{a) } X_1 = (1, 2, 1, 0) & Y_1 = (2, -1, 0, 1), \\ X_2 = (-1, 1, 1, 1), & Y_2 = (1, -1, 3, 7); \\ \text{b) } X_1 = (1, 2, -1, -2), & Y_1 = (2, 5, -6, -5), \\ X_2 = (3, 1, 1, 1), & Y_2 = (-1, 2, -7, -3), \\ X_3 = (-1, 0, 1, -1); & \\ \text{c) } X_1 = (1, 1, 0, 0), & Y_1 = (0, 0, 1, 1), \\ X_2 = (1, 0, 1, 1), & Y_2 = (0, 1, 1, 0). \end{array}$$

- 881 Find the coordinates of the vector  $X$  if the basis is  $E_1, E_2, E_3, E_4$ :

a)  $X = (1, 2, 1, 1), E_1 = (1, 1, 1, 1), E_2 = (1, 1, -1, -1),$   
 $E_3 = (1, -1, 1, -1), E_4 = (1, -1, -1, 1);$   
 b)  $X = (0, 0, 0, 1), E_1 = (1, 1, 0, 1), E_2 = (2, 1, 3, 1),$   
 $E_3 = (1, 1, 0, 0), E_4 = (0, 1, -1, -1).$

- 882 Find the matrix of coefficients and the formulas for the transformation of coordinates from the basis  $E_1, E_2, E_3, E_4$  to the basis  $E'_1, E'_2, E'_3, E'_4$ :

a)  $E_1 = (1, 0, 0, 0), E_2 = (0, 1, 0, 0), E_3 = (0, 0, 1, 0),$   
 $E_4 = (0, 0, 0, 1), E'_1 = (1, 1, 0, 0), E'_2 = (1, 0, 1, 0),$   
 $E'_3 = (1, 0, 0, 1), E'_4 = (1, 1, 1, 1);$   
 b)  $E_1 = (1, 2, -1, 0), E_2 = (1, -1, 1, 1),$   
 $E_3 = (-1, 2, 1, 1), E_4 = (-1, -1, 0, 1), E'_1 = (2, 1, 0, 1),$   
 $E'_2 = (0, 1, 2, 2), E'_3 = (-2, 1, 1, 2), E'_4 = (1, 3, 1, 2).$

- 883 A certain surface has the equation

$$x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1 \quad , \text{ in terms of the basis}$$

$E_1, E_2, E_3, E_4$ . Find the equation of the same surface with respect to the basis below, where the vectors are described by giving their coordinates in terms of the original basis:

$$E'_1 = (1, 1, 1, 1); \quad E'_2 = (1, 1, -1, -1); \quad E'_3 = (1, -1, 1, -1);$$

$$E'_4 = (1, -1, -1, 1) .$$

- \*884 This question concerns the space of polynomials in  $\cos x$  of degree  $\leq n$ . Give the formula for transformation of coordinates from the basis  $1, \cos x, \dots, \cos^n x$  to the basis  $1, \cos x, \dots, \cos nx$ , and the inverse transformation.
- 885 In four-dimensional space, find the equation of the line that passes through the origin and intersects each of the lines:

$$x_1 = 2 + 3t, \quad x_2 = 1 - t, \quad x_3 = -1 + 2t, \quad x_4 = 3 - 2t$$

$$x_1 = 7t, \quad x_2 = 1, \quad x_3 = 1 + t, \quad x_4 = -1 + 2t.$$

Find the respective points of intersection.

- 886 Show that every two lines in  $n$ -dimensional space can be embedded in a three-dimensional subspace.
- 887 (See problem 885) Find conditions on two lines, given in an  $n$ -dimensional space, so that a third line can be found that intersects each of the two given lines, and passes through the origin.
- 888 In  $n$ -dimensional space, show that every two planes can be embedded in a five-dimensional subspace.
- 889 Describe all possible ways that two planes can be arranged in  $n$ -dimensional space (see the preceding problem).

- 890 Show that a linear manifold can be characterized as the set of all vectors containing with every two vectors  $X_1, X_2$ , all linear combinations of the form  $\alpha X_1 + (1 - \alpha) X_2$ .

2. ELEMENTARY GEOMETRY OF  $n$ -DIMENSIONAL EUCLIDEAN SPACE.

- 891 Find the scalar product of the vectors  $X, Y$ :

a)  $X = (2, 1, -1, 2), \quad Y = (3, -1, -2, 1);$   
b)  $X = (1, 2, 1, -1), \quad Y = (-2, 3, -5, -1).$

- 892 Find the angle between the vectors  $X, Y$ :

a)  $X = (2, 1, 3, 2), \quad Y = (1, 2, -2, 1);$   
b)  $X = (1, 2, 2, 3), \quad Y = (3, 1, 5, 1);$   
c)  $X = (1, 1, 1, 2), \quad Y = (3, 1, -1, 0).$

- 893 Find the cosines of the angles between the line  $x_1 = x_2 = \dots = x_n$  and the coordinate axes.

- 894 Find the cosines of the interior angles of the triangle having as vertices:

$$A = (1, 2, 1, 2), \quad B = (3, 1, -1, 0), \quad C = (1, 1, 0, 1).$$

- 895 Find the length of the diagonal of an  $n$ -dimensional cube that has edges of length 1.

- 896 Find the number of diagonals of an  $n$ -dimensional cube that are orthogonal to a given diagonal.

- 897 In  $n$ -dimensional space, find  $n$  points with non-negative coordinates arranged so that every one is at distance 1 from the origin and every two are

one unit apart. Take the first point on the first coordinate axis, the second in the plane determined by the first two axes and so on. (These points, together with the origin, lie in the vertices of a regular simplex of edge-length 1.)

- 898 Find the coordinates of the center and the radius of the circumscribed sphere of the simplex defined in problem 897.
- 899 Normalize the vector  $(3, 1, 2, 1)$ .
- 900 Find a normalized vector orthogonal to each of the vectors  $(1, 1, 1, 1)$ ;  $(1, -1, -1, 1)$ ;  $(2, 1, 1, 3)$ .
- 901 The two vectors below are given. Supplement them by two others and normalize so that an orthonormal basis of the space is obtained.

$$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \quad \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{2}, -\frac{5}{6}\right)$$

- 902 Use the method of successive orthogonalization to find an orthogonal basis for the space spanned by the vectors  $(1, 2, 1, 3)$ ;  $(4, 1, 1, 1)$ ;  $(3, 1, 1, 0)$ .
- 903 Find two row vectors that are mutually orthogonal and orthogonal to each of the three rows of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 & -2 \\ 2 & 1 & -1 & 0 & 2 \end{pmatrix}$$

- 904 Give a geometric interpretation of the following system of homogeneous linear equations, and of a fundamental system of solutions of these equations in the  $n$ -dimensional space, the points of which have as coordinates the coefficient sets of the equations:

[illegible]

- 905 Find a fundamental orthonormal system of solutions  
for the system of equations:

$$\begin{aligned} 3x_1 - x_2 - x_3 + x_4 &= 0, \\ x_1 + 2x_2 - x_3 - x_4 &= 0. \end{aligned}$$

- 906 In each of the two problems below, resolve the vector  $\mathbf{X}$  into a sum of two vectors, one of which lies in the space spanned by the vectors  $\mathbf{A}_1, \mathbf{A}_2$  ( $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ ) and the other of which is orthogonal to this space.

a)  $X = (5, 2, -2, 2)$ ,  $A_1 = (2, 1, 1, -1)$ ,  $A_2 = (1, 1, 3, 0)$ ;  
 b)  $X = (-3, 5, 9, 3)$ ,  $A_1 = (1, 1, 1, 1)$ ,  $A_2 = (2, -1, 1, 1)$ ,  
 $A_3 = (2, -7, -1, -1)$ .

The first summand is called the orthogonal projection of  $X$ ; the second the orthogonal component of  $X$ .

- 907 Assume the vectors  $A_1, A_2, \dots, A_m$  are independent. Find a general expression for the orthogonal component of a vector  $X$  with respect to  $A_1, A_2, \dots, A_m$ .
- 908 Show that of all vectors in a given subspace  $P$ , the one that makes the smallest angle with the vector  $X$  is the orthogonal projection of  $X$  onto the subspace  $P$ .
- 909 Let the subspace  $P$  be spanned by the vectors  $A_1, A_2$  (or  $A_1, A_2, A_3$ ). Find the smallest possible angle between the given vector  $x$  and any vector in the space  $P$ .
- a)  $X = (1, 3, -1, 3), A_1 = (1, -1, 1, 1), A_2 = (5, 1, -3, 3);$   
b)  $X = (2, 2, -1, 1), A_1 = (1, -1, 1, 1), A_2 = (-1, 2, 3, 1), A_3 = (1, 0, 5, 3).$
- 910 Let an  $m$ -dimensional coordinate subspace be given; find the smallest possible angle made by any vector in the given subspace with the vector  $(1, 1, \dots, 1)$ .
- 911 Let a given subspace  $P$  be given and let  $Y$  run through the vectors of this subspace. Show that the vector  $X - Y$  has the smallest possible length



when  $Y$  is the vector  $X'$ , the orthogonal projection of  $X$  on  $P$  (this smallest possible length is called the distance from the point  $X$  to the subspace  $P$ ).

- 912 Find the distance from the point  $X$  to the linear manifold  $A_0 + t_1 A_1 + \cdots + t_m A_m$ :

- a)  $X = (1, 2, -1, 1)$ ,  $A_0 = (0, -1, 1, 1)$ ,  $A_1 = (0, -3, -1, 5)$ ,  
 $A_2 = (4, -1, -3, 3)$ ;  
 b)  $X = (0, 0, 0, 0)$ ,  $A_0 = (1, 1, 1, 1)$ ,  $A_1 = (1, 2, 3, 4)$ .

- 913 In the space of polynomials with real coefficients and degree less than or equal to  $n$ , the scalar product of the polynomials  $f_1, f_2$  is defined as
- $$\int_0^1 f_1(x) f_2(x) dx \quad . \quad \text{Find the distance from the origin}$$
- to the manifold determined by the polynomial

$$x^n + a_1 x^{n-1} + \cdots + a_n \quad .$$

- 914 Give a procedure for finding the shortest distance between two linear manifolds  $X_0 + P, Y_0 + Q$ , that is between two points in these respective manifolds.
- 915 See problem 897. The vertices of an  $n$ -dimensional regular simplex with edge length 1 are separated into two collections of  $m + 1$  and  $n - m$  vertices, respectively. Through each collection, a linear manifold is drawn with minimum possible dimension. Find the shortest distance between the points of these manifolds, and locate the points between which this distance is realized.

\*916 Two planes are determined by the pairs of vectors  $A_1, A_2; B_1, B_2$  in four-dimensional space. Find the smallest possible angle formed between pairs of vectors, one from the first plane and one from the second.

$$a) A_1 = (1, 0, 0, 0), A_2 = (0, 1, 0, 0), B_1 = (1, 1, 1, 1), \\ B_2 = (2, -2, 5, 2);$$

$$b) A_1 = (1, 0, 0, 0), A_2 = (0, 1, 0, 0), B_1 = (1, 1, 1, 1), \\ B_2 = (1, -1, 1, -1).$$

\*917 A three-dimensional hyperplane is drawn through the center of a four-dimensional cube and orthogonal to a diagonal. Describe the solid of intersection.

\*918 If  $B_1, B_2, \dots, B_m$  is a set of  $m$  linearly independent vectors, the parallelipipedon determined by these vectors is defined as the set of all points of the form

$$t_1 B_1 + t_2 B_2 + \dots + t_m B_m, 0 \leq t_1 \leq 1, \dots, 0 \leq t_m \leq 1.$$

Define inductively the volume of the parallelipipedon as the product of the volume of the base

$[B_1, B_2, \dots, B_{m-1}]$  by the altitude, the latter being the distance from the vector  $B_m$  to the space spanned by the base. To begin the induction define the volume of a parallelipipedon  $[B_1]$  consisting of a single vector as the length of the vector.

a) Give a formula for the square of the volume and deduce that the value of the volume is independent of the ordering of the vertices.

b) Show that  $V[cB_1, B_2, \dots, B_m] = |c| \cdot V[B_1, B_2, \dots, B_m]$ .

c) Show that  $V[B'_1 + B''_1, B_2, \dots, B_m] \leq V[B'_1, B_2, \dots, B_m] + V[B''_1, B_2, \dots, B_m]$ , and find when the inequality degenerates to an equality.

919 Show that the volume of an  $n$ -dimensional parallelipipedon in  $n$ -dimensional space is the same as the absolute value of the determinant of the matrix whose rows are the coordinates of the generating vectors.

\*920 Let  $C_1, C_2, \dots, C_m$  be the orthogonal projections of the vectors  $B_1, B_2, \dots, B_m$  onto some space. Prove that

$$V[C_1, C_2, \dots, C_m] \leq V[B_1, B_2, \dots, B_m].$$

\*921 See problem 518. Show that

$$V[A_1, A_2, \dots, A_m, B_1, \dots, B_k] \leq V[A_1, \dots, A_m] \cdot V[B_1, \dots, B_k]$$

922 See problem 519. Show that

$$V[A_1, A_2, \dots, A_m] \leq |A_1| \cdot |A_2| \dots |A_m|$$

923 Find the volume of the  $n$ -dimensional sphere by using Cavalieri's principle.

924 The scalar product of two polynomials of degree  $\leq n$  is defined as  $\int_0^1 f_1(x) f_2(x) dx$ . Find the volume of the parallelipipedon generated by vectors defined as the sequence of coefficients of a set of polynomials.

3. PROPER VALUES AND PROPER VECTORS OF A MATRIX.

925 Find the proper values and proper vectors of the following matrices:

$$\begin{aligned} & \text{a) } \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}; \quad \text{b) } \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}; \quad \text{c) } \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}; \quad \text{d) } \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}; \\ & \text{e) } \begin{pmatrix} 5 & 6 & -3 \\ -1 & 0 & 1 \\ 1 & 2 & -1 \end{pmatrix}; \quad \text{f) } \begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix}; \quad \text{g) } \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \\ & \text{h) } \begin{pmatrix} 0 & 2 & 1 \\ -2 & 0 & 3 \\ -1 & -3 & 0 \end{pmatrix}; \quad \text{i) } \begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ 4 & -8 & -2 \end{pmatrix}; \quad \text{j) } \begin{pmatrix} 2 & 5 & -6 \\ 4 & 6 & -9 \\ 3 & 6 & -8 \end{pmatrix}. \end{aligned}$$

926 Find the connection between the proper values of the matrix  $A$ , and those of its inverse  $A^{-1}$ .

927 Find the relation between the proper values of the matrix  $A$ , and those of its square  $A^2$ .

928 Find the proper values of the matrix  $A^m$  in terms of the proper values of the matrix  $A$ .

929 Let  $f(\lambda)$  be the characteristic polynomial of an  $n$ -th order matrix  $A$ . Let  $f(x) = b_0(x - \xi_1)(x - \xi_2) \dots (x - \xi_m)$ . Find the determinant of the matrix  $f(A)$ .

930 Let  $f(x)$  be a given polynomial. Evaluate the determinant of the matrix  $f(A)$ , in terms of the proper values of the matrix  $A$ .

- 931 Use the result of the preceding problem to find the proper values of the matrix  $f(A)$  in terms of the proper values of the matrix  $A$ .
- 932 Show that every proper vector of the matrix  $A$  is also a proper vector of the matrix  $f(A)$ , where  $f$  is any polynomial.
- \*933 Find the proper values of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \varepsilon & \varepsilon^2 & \dots & \varepsilon^{n-1} \\ 1 & \varepsilon^2 & \varepsilon^4 & \dots & \varepsilon^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \varepsilon^{n-1} & \varepsilon^{2(n-1)} & \dots & \varepsilon^{(n-1)^2} \end{pmatrix},$$

where  $\varepsilon = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$ , and  $n$  is odd.

- \*934 With  $\varepsilon$  defined as in the preceding problem, evaluate the sum

$$1 + \varepsilon + \varepsilon^4 + \dots + \varepsilon^{(n-1)^2}.$$

- 935 Find the proper values of the following matrices:

a)  $\begin{pmatrix} 0 & x & x & \dots & x \\ y & 0 & x & \dots & x \\ y & y & 0 & \dots & x \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y & y & y & \dots & 0 \end{pmatrix};$  b)  $\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ a_n & a_1 & \dots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \dots & a_1 \end{pmatrix};$

c)  $\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & -1 & 0 & \ddots & \\ & & & \ddots & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix},$

- \*936 See problem 534. Show how the proper values of the Kronecker product of two matrices can be found in terms of the proper values of the factors.
- 937 Let  $A, B$  be arbitrary square matrices. Show that  $AB$  and  $BA$  have the same characteristic polynomial.
- 938 Let  $A, B$  be rectangular matrices with  $m$  rows and  $n$  columns [ $n$  rows and  $m$  columns] respectively,  $n > m$ . Show that the characteristic polynomials of  $AB, BA$  differ only in the factor  $(-\lambda)^{n-m}$ .

Reference: H. Flanders, Elementary Divisors of  $AB$  and  $BA$ , Proceedings of the American Mathematical Society, 2 (1951) 871-874.

#### 4. QUADRATIC FORMS AND SYMMETRIC MATRICES

- 939 Transform each of the following into a sum of squares:

- a)  $x_1^2 + 2x_1x_2 + 2x_2^2 + 4x_2x_3 + 5x_3^2$ ;
- b)  $x_1^2 - 4x_1x_2 + 2x_1x_3 + 4x_2^2 + x_3^2$ ;
- c)  $x_1x_2 + x_2x_3 + x_3x_1$ ;
- d)  $x_1^2 - 2x_1x_2 + 2x_1x_3 - 2x_1x_4 + x_2^2 + 2x_2x_3 - 4x_2x_4 + x_3^2 - 2x_4^2$ ;
- e)  $x_1^2 + x_1x_2 + x_3x_4$ .

$$f_k(x_{k+1}, \dots, x_n) = f - a'_1 x_1'^2 - \dots - a_k x_k'^2$$





$$D_f \leq a_{11} D_\varphi$$

holds.

- 947 Let  $l_1, l_2, \dots, l_p, l_{p+1}, l_{p+2}, \dots, l_{p+q}$  be real linear forms in  $x_1, x_2, \dots, x_n$ . Show that the number of squares in the canonical representation of the form

$$\begin{aligned} f(x_1, x_2, \dots, x_n) &= \\ &= l_1^2 + l_2^2 + \dots + l_p^2 - l_{p+1}^2 - l_{p+2}^2 - \dots - l_{p+q}^2 \end{aligned}$$

cannot exceed  $p$ , and the number of negative squares cannot exceed  $q$ .

- \*948 Let  $s_0, s_1, \dots$  be the sums of the powers of the roots of the equation  $x^n + a_1 x^{n-1} + \dots + a_n = 0$ , the coefficients of which are real. Show that the number of negative squares in the canonical representation of the quadratic form  $\sum_{i,k=1}^n s_{i+k-2} x_i x_k$  is equal to the number of pairs of complex conjugate roots of the given equation.

Prove the following theorems:

- 949 If all roots of a polynomial equation with real coefficients are real and distinct, then the inequalities

$$\begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix} > 0; \begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix} > 0; \dots; \Delta = \begin{vmatrix} s_0 & s_1 & \dots & s_{n-1} \\ s_1 & s_2 & \dots & s_n \\ \dots & \dots & \dots & \dots \\ s_{n-1} & s_n & \dots & s_{2n-2} \end{vmatrix} > 0.$$

will hold, and conversely.

\*950 If the forms

[illegible]

are non-negative definite, then the form

[illegible]

will be non-negative definite.

951 Use an orthogonal transformation to transform each  
of the following into canonical form:

- $2x_1^2 + x_2^2 - 4x_1x_2 - 4x_2x_3;$
- $x_1^2 + 2x_2^2 + 3x_3^2 - 4x_1x_2 - 4x_2x_3;$
- $3x_1^2 + 4x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_2x_3;$
- $2x_1^2 + 5x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_1x_3 - 8x_2x_3;$
- $x_1^2 - 2x_2^2 - 2x_3^2 - 4x_1x_2 + 4x_1x_3 + 8x_2x_3;$
- $5x_1^2 + 6x_2^2 + 4x_3^2 - 4x_1x_2 - 4x_1x_3;$
- $3x_1^2 + 6x_2^2 + 3x_3^2 - 4x_1x_2 - 8x_1x_3 - 4x_2x_3;$

- h)  $7x_1^2 + 5x_2^2 + 3x_3^2 - 8x_1x_2 + 8x_2x_3$ ;  
 i)  $2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_4^2 - 4x_1x_2 + 2x_1x_4 + 2x_2x_3 - 4x_3x_4$ ;  
 j)  $2x_1x_2 + 2x_3x_4$ ;  
 k)  $x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2x_1x_2 - 2x_1x_4 - 2x_2x_3 + 2x_3x_4$ ;  
 l)  $2x_1x_2 + 2x_1x_3 - 2x_1x_4 - 2x_2x_3 + 2x_2x_4 + 2x_3x_4$ ;  
 m)  $x_1^2 + x_2^2 + x_3^2 + x_4^2 - 2x_1x_2 + 6x_1x_3 - 4x_1x_4$   
      $- 4x_2x_3 + 6x_2x_4 - 2x_3x_4$ ;  
 n)  $8x_1x_3 + 2x_1x_4 + 2x_2x_3 + 8x_2x_4$ .

952 Use an orthogonal transformation to transform each of the following into canonical form:

$$a) \sum_{i=1}^n x_i^2 + \sum_{i < k} x_i x_k; \quad b) \sum_{i < k} x_i x_k.$$

953 Use an orthogonal transformation to transform the following into canonical form:

$$x_1x_2 + x_2x_3 + \dots + x_{n-1}x_n.$$

954 Suppose that all proper values of the real matrix  $A$  lie in the segment  $[a, b]$ . Show that the quadratic form having matrix  $A - \lambda E$  is negative definite for  $\lambda > b$  and positive definite for  $\lambda < a$ . The converse is also true.

955 Suppose that all the proper values of the real symmetric matrices  $a, b$  lie on the segments  $[a, c]$ ;  $[b, d]$  respectively. Show that all the

proper values of the matrix  $A + B$  lie in the segment  $[a + b, c + d]$ .

956 Let  $A$  be a real matrix,  $A'$  its transpose. By the norm of  $A$  (denoted  $||A||$ ) we mean the positive square root of the greatest proper value of the matrix  $A'A$ . Establish the following

a)  $||A|| = ||A'||$ ;

b)  $\text{Length}(AX) \leq ||A|| \cdot \text{Length}(X)$ , and there is a vector  $X_0$  for which equality occurs;

c)  $||A + B|| \leq ||A|| + ||B||$ ;

d)  $||AB|| \leq ||A|| \cdot ||B||$ ;

e) Every proper value of the matrix  $A$  is less than or equal to  $||A||$  in absolute value.

957 Show how to represent an arbitrary real nonsingular matrix as the product of an orthogonal matrix by a triangular matrix of the form

$$\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ & b_{22} & \dots & b_{2n} \\ & & \ddots & \vdots \\ & & & b_{nn} \end{pmatrix};$$

in the latter the diagonal elements  $b_{ii}$  are positive. This representation is unique.



## 5. LINEAR SPACES. THE JORDAN CANONICAL FORM.

962 Let  $A$  be a linear transformation defined on a space of dimension  $n$ . Show that the dimension of the range of  $A$  (the set of all vectors  $y$  of the form  $y = Ax$ ) is equal to the rank of  $A$ .

963 Let  $Q$  be a  $q$ -dimensional subspace of the  $n$ -dimensional space  $R$ ; let  $A$  be a linear transformation of  $R$  of rank  $r$ ; let  $Q'$  be the image of  $Q$ . If  $q'$  is the dimension of  $Q'$ , show that the following relation holds:

$$q + r - n \leq q' \leq \min(q, r).$$

964 Use the result of problem 963 to show that the rank  $\rho$  of the product of two matrices of ranks  $r_1, r_2$  satisfies the inequalities

$$r_1 + r_2 - n \leq \rho \leq \min(r_1, r_2).$$

\*965 Let  $Q, Q'$  be arbitrary complementary subspaces of the space  $P$ . Then every vector  $X \in P$  can be represented uniquely as the sum of vectors  $Y \in Q, Y' \in Q'$ . For each vector  $X$  a linear transformation can be defined carrying  $X$  into its component  $Y$  in  $Q$ ; this transformation is called a projection onto  $Q$  parallel to  $Q'$ . Prove that this transformation is a linear transformation and show that the matrix of this transformation satisfies the relation  $A^2 = A$ .

Does this last assertion depend on the particular basis chosen? Conversely, if  $A$  is a linear transformation satisfying the relation  $A^2 = A$ , then it corresponds to a projection operator.

- \*966 A projection is called orthogonal if the complementary spaces  $Q'$ ,  $Q$  are orthogonal. Show that the matrix of a projection is symmetric whenever the base is orthonormal. Conversely every symmetric idempotent matrix ( $A^2 = A$ ) is the matrix of an orthogonal projection.
- \*967 Show that every non-null proper value of a skew-symmetric matrix is pure imaginary; the real and imaginary parts of the corresponding proper vector have the same length and are mutually orthogonal.
- \*968 Show that the skew-symmetric matrix  $A$  can be transformed by a suitable orthogonal matrix  $P$  as follows:

$$P^{-1}AP = \begin{pmatrix} 0 & a_1 & & & & \\ -a_1 & 0 & & & & \\ & & 0 & a_2 & & \\ & & -a_2 & 0 & & \\ & & & & \ddots & \\ & & & & & 0 & a_k \\ & & & & & -a_k & 0 \\ & & & & & & & 0 \\ & & & & & & & & \ddots \\ & & & & & & & & & 0 \end{pmatrix}.$$

669 Establish the following theorem: Let  $A$  be a skew-symmetric matrix. Then the matrix  $(E - A)(E + A)^{-1}$  is orthogonal and does not have  $-1$  as a proper value. Conversely every orthogonal matrix which does not have  $-1$  as a proper value can be written in this form.

670 Show that every proper value of an orthogonal matrix has absolute value 1.

671 Show that every proper vector of an orthogonal matrix corresponding to a nonreal proper value can be written in the form  $X + iY$ , where  $X, Y$  are real have the same length, and are orthogonal.

672 Show that every orthogonal matrix can be written in the form  $Q^{-1}TQ$ , where  $Q$  is orthogonal, and  $T$  has the form

$$\begin{array}{ccccccc}
 \cos \varphi_1 & -\sin \varphi_1 & & & & & \\
 \sin \varphi_1 & \cos \varphi_1 & & & & & \\
 & \cos \varphi_2 & -\sin \varphi_2 & & & & \\
 & \sin \varphi_2 & \cos \varphi_2 & & & & \\
 & & . & & & & \\
 & & . & & & & \\
 & & . & & & & \\
 & & & 1 & & & \\
 & & & & 1 & & \\
 & & & & & . & \\
 & & & & & . & \\
 & & & & & . & \\
 & & & & & & 1 \\
 & & & & & & -1 \\
 & & & & & & . \\
 & & & & & & . \\
 & & & & & & . \\
 & & & & & & -1 \\
 & & & & & & -1
 \end{array}$$



973 Transform each of the following matrices into Jordan canonical form:

$$\text{a) } \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & -1 \end{pmatrix}; \quad \text{b) } \begin{pmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{pmatrix};$$

$$\text{c) } \begin{pmatrix} 13 & 16 & 16 \\ -5 & -7 & -6 \\ -6 & -8 & -7 \end{pmatrix}; \quad \text{d) } \begin{pmatrix} 3 & 0 & 8 \\ 3 & -1 & 6 \\ -2 & -0 & -5 \end{pmatrix};$$

$$\text{e) } \begin{pmatrix} -4 & 2 & 10 \\ -4 & 3 & 7 \\ -3 & 1 & 7 \end{pmatrix}; \quad \text{f) } \begin{pmatrix} 7 & -12 & -2 \\ 3 & -4 & 0 \\ -2 & 0 & -2 \end{pmatrix};$$

$$\text{g) } \begin{pmatrix} -2 & 8 & 6 \\ -4 & 10 & 6 \\ 4 & -8 & -4 \end{pmatrix}; \quad \text{h) } \begin{pmatrix} 0 & 3 & 3 \\ -1 & 8 & 6 \\ 2 & -14 & -10 \end{pmatrix};$$

$$\text{i) } \begin{pmatrix} -1 & 1 & 1 \\ -5 & 21 & 17 \\ 6 & -26 & -21 \end{pmatrix}; \quad \text{j) } \begin{pmatrix} 8 & 30 & -14 \\ -6 & -19 & 9 \\ -6 & -23 & 11 \end{pmatrix};$$

$$\text{k) } \begin{pmatrix} 4 & 5 & -2 \\ -2 & -2 & 1 \\ -1 & -1 & 1 \end{pmatrix}; \quad \text{l) } \begin{pmatrix} 3 & 7 & -3 \\ -2 & -5 & 2 \\ -4 & -10 & 3 \end{pmatrix};$$

$$\text{m) } \begin{pmatrix} 9 & 22 & -6 \\ -1 & -4 & 1 \\ 8 & 16 & -5 \end{pmatrix}; \quad \text{n) } \begin{pmatrix} 1 & -1 & 2 \\ 3 & -3 & 6 \\ 2 & -2 & 4 \end{pmatrix}; \quad \text{o) } \begin{pmatrix} 1 & 1 & -1 \\ -3 & -3 & 3 \\ -2 & -2 & 2 \end{pmatrix}.$$

- 974 Transform each of the following matrices into Jordan canonical form:

$$\text{a) } \begin{pmatrix} 3 & 1 & 0 & 0 \\ -4 & -1 & 0 & 0 \\ 7 & 1 & 2 & 1 \\ -17 & -6 & -1 & 0 \end{pmatrix}; \quad \text{b) } \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \text{c) } \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

- \*975 Let the matrix  $A$  have finite order:  $A^m = E$  for some positive integer  $m$ . Show that the canonical form of  $A$  is diagonal.
- \*976 See problem 531. Find the proper values of the matrix  $A_{(m)}$  in terms of the proper values of the matrix  $A$ .
- 977 Show how to transform an arbitrary matrix  $A$  into its transpose.
- \*978 Show that every matrix can be transformed into the product of two symmetric matrices, one of which is nonsingular.
- 979 Starting with a given  $n$ -th order matrix  $A$ , construct a sequence of matrices according to the following rule:

$$\begin{aligned}
A_1 &= A; & \text{Sp} A_1 &= p_1; & A_1 - p_1 E &= B_1, \\
B_1 A &= A_2; & \frac{1}{2} \text{Sp} A_2 &= p_2; & A_2 - p_2 E &= B_2, \\
B_2 A &= A_3; & \frac{1}{3} \text{Sp} A_3 &= p_3; & A_3 - p_3 E &= B_3, \\
&\dots\dots\dots \\
B_{n-1} A &= A_n; & \frac{1}{n} \text{Sp} A_n &= p_n; & A_n - p_n E &= B_n.
\end{aligned}$$

Show that the numbers  $p_1, p_2, \dots, p_n$  are the coefficients of the characteristic polynomial

$$(-1)^n [\lambda^n - p_1 \lambda^{n-1} - p_2 \lambda^{n-2} - \dots - p_n]$$

of  $A$ . Show that the matrix  $B_n$  is the null matrix.

Finally show that if  $A$  is nonsingular, then the relation  $\frac{1}{p_n} B_{n-1} = A^{-1}$  holds.

- 980 Let  $C$  be a matrix for which the equation  $XY - YX = C$  has a solution in square matrices  $X, Y$ . Then the trace (the sum of the diagonal elements) of  $C$  is zero. Prove the converse of this theorem.
- 981 Let  $C$  be a matrix of complex elements,  $C^*$  its hermitian conjugate (conjugate of the transpose). If all singular values  $\rho_i$  of  $C$  satisfy  $|\rho_i| < 1$ , show that a positive definite hermitian matrix  $H$  can be found  $[H = H^*; x \neq 0 \rightarrow x H x^* > 0]$  such that  $H - CHC^* = I$ , identity matrix.

A singular value of  $C$  is a proper value of  $CC^*$ .

- 982 If  $C$  is the matrix of problem 981, and if  $B$  is an arbitrary positive definite hermitian matrix  $[B = B^*; x \neq 0 \rightarrow x B x^* > 0]$ , a positive definite hermitian matrix  $H$  can be found such that

$$H - CHC^* = B.$$

- 983 Problem 982 uses the (strong) hypothesis that all proper values of  $CC^*$  have modulus less than 1. In point of fact, the conclusion remains valid under the (weaker) hypothesis that all proper values of  $C$  have modulus less than 1. Assume this proved, and from it, establish the Lyapunov theorem: If  $R$  is an arbitrary positive-definite hermitian matrix and  $A$  is a stable matrix (i.e. a matrix, all proper values of which have negative real part), there is a positive-definite hermitian matrix  $G$  such that

$$AG + GA^* = -R.$$



CHAPTER I - HINTS  
COMPLEX NUMBERS

- 11 See problem 10.
- 13 First establish the theorem for each of the four binary operations; then use mathematical induction.
- 18 Assume that the left member is the sum of two squares.
- 27 Set  $x = a + bi$ ,  $y = c + di$ .
- 28 Set  $z = \cos \varphi + i \sin \varphi$ .
- 31 Set  $z = t^2$ ,  $z' = t'^2$ . Use problem 27.
- 37 Use polar form.
- 38  $1 + \omega = -\omega^2$ .
- 40 Use the half angle formulas.
- 41 Use Demoivre's theorem and note the following
- $$z = \cos \theta \pm i \sin \theta; \quad \frac{1}{z} = \cos \theta \mp i \sin \theta.$$
- 51 Let  $\alpha = \cos x + i \sin x$ . Then
- $$\cos^{2m} x = \left( \frac{\alpha + \alpha^{-1}}{2} \right)^{2m}$$
- 52 Show that the coefficient of  $(2 \cos x)^{m-2p}$  is  $(-1)^p (C_p^{m-p} + C_{p-1}^{m-p-1})$ . Use mathematical induction.
- 53 This problem is similar to the preceding.

- 54 Use Newton's formula for the binomial expansion of  $(1 + i)^n$ .
- 55 Similar to the preceding; choose the proper binomial.
- 56 In this case the binomial to choose is  $\left(1 + i\frac{\sqrt{3}}{3}\right)^n$ .
- 68 Show that the problem comes down to the calculation of the limit of the sum  $1 + \alpha + \alpha^2 + \dots$ , where  $\alpha = \frac{-1 + i}{2}$ .
- 69 Use the formula  $\sin^2 \alpha = \frac{1}{2} - \frac{\cos 2\alpha}{2}$ .
- 71 Use the formulas
- $$\cos^3 \alpha = \frac{\cos 3\alpha}{4} + \frac{3 \cos \alpha}{4}; \quad \sin^3 \alpha = \frac{3 \sin \alpha}{4} - \frac{\sin 3\alpha}{4}.$$
- 72 To calculate sums of the form  $1 + 2a + 3a^2 + 4a^3 + \dots + na^{n-1}$ ,  $1 + 2^2a + 3^2a^2 + \dots + n^2a^{n-1}$  it is convenient to multiply first by the factor  $1 - a$ .
- 76  $x_1 = \alpha + \beta; \quad x_2 = \alpha\omega + \beta\omega^2; \quad x_3 = \alpha\omega^2 + \beta\omega;$   
 $\alpha^3 + \beta^3 = -q; \quad 3\alpha\beta = -p.$
- 77 Multiply by -27 and think of the left member as the discriminant of some cubic equation.
- 78 Set  $x = \alpha + \beta$ .
- 87 Use the fact that  $\epsilon^n = -1$ .
- 88 If  $\epsilon = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$ , note that the sum in question has the form  $1 + \epsilon + \epsilon^2 + \dots + \epsilon^{n-1}$ .

- 89 Consider two cases: 1)  $k$  divides  $n$ ; 2)  $k$  does not divide  $n$ .
- 91, 92 Multiply by  $1 - \epsilon$ .
- 94 a) Start with all 15 roots of unity; subtract the sum of the roots of degrees 1, 3, 5.
- 97 If a 14-sided figure is inscribed on a circle of radius 1, the length of its side is  $2 \sin \pi/14$ .  
Use the additional fact that the equation  $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$  has the number  $\cos 4\pi/7 + i \sin 4\pi/7$  as one root.
- 98 1) If  $x_1, x_2, \dots, x_n$  are roots of the equation  $a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0$ , then  $a_0 x^n + a_1 x^{n-1} + \dots + a_n = a_0 (x - x_1) \dots (x - x_n)$ ;  
2) If  $\epsilon$  is an  $n$ -th root of 1, then the complex conjugate is also an  $n$ -th root of 1.
- 99 Use the result of problem in 98 in the special case  $x = 1$ .
- 100 Use the identity showing the factorization of  $x^n - 1$  into first degree factors.
- 101 Factor  $x^n - 1$  into factors of the first degree; then set  
1)  $x = \cos \theta + i \sin \theta$ ;  
2)  $x = \cos \theta - i \sin \theta$ .
- 103 Use the fact that two conjugate complex numbers have the same modulus.



- 105 a) Transform the equation into the form

$$\left(\frac{x+1}{x-1}\right)^m = 1.$$

- 107 Let

$$S = \cos \varphi + C_1^n \cos(\varphi + \alpha) x + \dots + \cos(\varphi + n\alpha) x^n,$$

$$T = \sin \varphi + C_1^n \sin(\varphi + \alpha) x + \dots + \sin(\varphi + n\alpha) x^n.$$

Calculate  $S + Ti$ ,  $S - Ti$  and determine  $S$  from the results of the calculation.

- 113 First show that  $\varphi(p^\alpha) = p^\alpha \left(1 - \frac{1}{p}\right)$ , if  $p$  is a prime number. This can be done by counting up all the positive integers less than  $p^\alpha$  and not divisible by  $p$ . Then use 112.

- 116 Show that the only primitive roots of  $x^{p^m} - 1$  are those that are not simultaneously roots of  $x^{p^{m-1}} - 1$ .

- 117 Show that if  $n$  is odd, every primitive  $2n$ -th root of unity is the negative of a primitive  $n$ -th root of  $-1$ ; there are no others.

- 119 Use 118.

- 120 Use 115, 116, 111 and establish the following

- 1)  $\mu(p) = -1$ , when  $p$  is prime;
- 2)  $\mu(p^\alpha) = 0$ , when  $p$  is prime and  $\alpha > 1$ ;
- 3)  $\mu(ab) = \mu(a)\mu(b)$ , if  $a, b$  are relatively prime.

- 122 Show that if  $\epsilon_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$

corresponds to the value of  $n_1$ , then  $x - \epsilon_k$

appears in the right member of the equation to the power  $\sum \mu(d_1)$ , where  $d_1$  runs through all divisors of  $n/n_1$ .

123 Consider two cases: 1)  $n$  is a power of a prime number; 2)  $n$  is the product of powers of different primes. For case 1 use problem 116; for case 2 problems 119, 122.

124 Consider four cases:

- 1)  $n$  odd greater than 1;
- 2)  $n$  a power of 2;
- 3)  $n = 2n_1$ , where  $n_1$  is odd and greater than 1;
- 4)  $n = 2^k n_1$ , where  $k$  is greater than 1,  $n_1$  is odd and greater than 1.

125 Use the identity

$$\begin{aligned} x_1 x_2 + x_1 x_3 + \dots + x_{n-1} x_n &= \\ &= \frac{(x_1 + x_2 + \dots + x_n)^2 - (x_1^2 + x_2^2 + \dots + x_n^2)}{2}. \end{aligned}$$

Separate into three cases:

- 1)  $n$  odd;
- 2)  $n$  twice an odd number;
- 3)  $n = 2^k n_1$ , where  $k$  greater than 1,  $n_1$  odd.

126 Multiply the sum  $S$  by its conjugate and note that  $\varepsilon^{x^2}$  does not change if  $x$  is replaced by  $x + n$ .

CHAPTER II - HINTS  
COMPUTATION OF DETERMINANTS

- 132 Every pair of numbers is reversed.
- 133 The number of inversions in the second permutation is equal to the number of sequences in the first in natural order.
- 145 Show that every term contains the factor 0.
- 149, 150 Interchange rows and columns.
- 153 Note how the value of the determinant changes if an arbitrary permutation is performed on the columns
- 154 a) Use the factor theorem and note that, if  $x = a_1$ , the determinant has two equal rows.
- 155 Add 100 times the first column plus 10 times the second column to the third column.
- 156 First subtract the first column from each of the others.
- 163 Subtract the first column from the second.
- 179 Add the sum of the last  $n - 1$  rows to the first.
- 180,
- 181, Subtract the first row from each of the others.
- 182
- 183 Subtract the second row from each of the others.
- 184 Add the first row to the second.
- 185 Add the sum of all the remaining columns to the first.

- 186, Modify the first column as follows: subtract the  
187 second, add the third, etc.
- 188 Expand according to the minors of the elements of  
the first column; or add to the last column  
 $x^n$  times the first column +  $x^{n-1}$  times the second  
column + ...
- 189 Add  $x^{n-1}$  times the first column +  $x^{n-2}$  times the  
second column + ... to the last column.
- 190 First set up the matrix whose determinant is  
needed; then proceed as follows: add to the last  
column,  $-1$  times the first,  $-x$  times the second,  
 $-x^2$  times the third, etc.
- 191 Subtract  $a_1$  times the last column from the first;  
 $a_2$  times the last column from the second; ...  
 $a_n$  times the last column from the  $n$ -th.
- 192 Add the sum of the remaining columns to the first.
- 194 Add the sum of the remaining columns to the last.
- 195 Factor  $a_1$  from the first column,  $a_2$  from the  
second,  $a_3$  from the third, ...,  $a_n$  from the last.  
Then add all the columns to the last column.
- 196 Factor  $h$  from the first column; add the first  
column to the second.
- 197 Multiply the first row and the first column by  $x$ ;  
add all the remaining rows to the first; make  
further transformations of a similar type.

- 198 To the second row add  $-a_1$  times the first; to the third row add  $-a_2$  times the first;  $\dots$ . Perform a similar transformation on the columns.
- 199, Add the remaining columns to the first.  
200
- 201 First add  $-a$  times the next last column to the last; then add  $-a$  times the preceding column to the next last column;  $\dots$ .
- 202 Similar to 201 using the rows; the factor  $a$  is replaced by 1. Finally add the first column to each of the others.
- 203 Multiply the first row by  $b_0$ , the second by  $b_1$  and so on; finally add the sum of the last  $n - 1$  rows to the first.
- 204 Factor  $a$  from the first row; subtract the first row from the second.
- 205 Expand by minors of the elements of the first row.
- 206 Write as the sum of 2 determinants.
- 208 Imagine 0 as the third summand of every term not on the principal diagonal; write the determinant as the sum of  $2^n$  determinants. Use problems 206, 207.
- 211 Multiply the first column by  $x^{n-1}$ , the second by  $x^{n-2}$  and so on.

- 212 Expand according to minors of the last column, and note the inductive relation

$$\Delta_n = x_n \Delta_{n-1} + a_n x_1 x_2 \dots x_{n-1}$$

where  $\Delta_n$  is the value of the  $n$ -th order determinant. Finally use mathematical induction.

- 213 Expand according to minors of the elements of the last row and show that  $\Delta_{n+1} = x_n \Delta_n + a_n y_1 y_2 \dots y_n$ . See the hint for number 212.

- 214 Factor  $a_1$  from the second column;  $a_2$  from the third;  $\dots$ ;  $a_n$  from the  $(n+1)$ -st. Reverse the sign of the first column and add all the remaining columns to the first.

- 215 Expand according to minors of the elements of the first row.

- 216 Expand according to minors of the elements of the first row and show that

$$\Delta_n = a_1 a_2 \dots a_{n-1} - \Delta_{n-1}.$$

See the hint to problem 212.

- 219 Use the result of problem 217.

- 221 Expand according to the elements of the first row and note that

$$\Delta_n = x \Delta_{n-1} - \Delta_{n-2}.$$

Use the hint to problem 212.

- 222 Subtract  $y_n/y_{n-1}$  times the next last row from the last. Establish the recursion formula

$$\Delta_n = \frac{y_n}{y_{n-1}}(x_n y_{n-1} - x_{n-1} y_n) \Delta_{n-1}.$$

See the hint to number 212.

- 223 Express as the sum of two determinants and establish the recursion formula

$$\Delta_n = a_n \Delta_{n-1} + a_1 a_2 \dots a_{n-1}.$$

- 225 Express as the sum of two determinants and establish the recursion formula

$$\Delta_n = (a_n - x) \Delta_{n-1} + x(a_1 - x) \dots (a_{n-1} - x).$$

- 226 Rewrite  $x_n$  in the form  $x_n = (x_n - a_n) + a_n$ ; write the determinant as the sum of two determinants and establish the recursion formula

$$\Delta_n = (x_n - a_n) \Delta_{n-1} + a_n (x_1 - a_1)(x_2 - a_2) \dots (x_{n-1} - a_{n-1}).$$

- 227 Write as the sum of two determinants and establish the recursion formula

$$\Delta_n = (x_n - a_n b_n) \Delta_{n-1} + a_n b_n (x_1 - a_1 b_1) \dots (x_{n-1} - a_{n-1} b_{n-1}).$$

- 228 Write as the sum of two determinants and establish the recursion formula

$$\Delta_n = -m \Delta_{n-1} + (-1)^{n-1} m^{n-1} x_n.$$

- 230 Expand according to minors of the elements of the first row and show that

$$\Delta_{2n} = (a^2 - b^2) \Delta_{2n-2}.$$

- 231 From every row subtract the preceding; and add all the remaining rows to the second one. Finally expand the determinant according to minors of the elements of the last row and establish the recursion formula

$$\Delta_n = [a + (n-1)b] \Delta_{n-1} + a(a+b) \dots [a + (n-2)b].$$

See the hint for problem 212.

- 232 Write as the sum of two determinants and establish the recursion formula

$$\Delta_n = x(x - 2a_n) \Delta_{n-1} + a_n^2 x^{n-1} \prod_{i=1}^{n-1} (x - 2a_i).$$

- 233 Make the algebraic transformation  $(x - a_n)^2 = x(x - 2a_n) + a_n^2$ , write the determinant as the sum of two determinants and establish the recursion formula

$$\Delta_n = x(x - 2a_n) \Delta_{n-1} + a_n^2 x^{n-1} (x - 2a_1) \dots (x - 2a_{n-1}).$$

- 234 Write as the sum of two determinants and establish the recursion formula

$$\Delta_n = \Delta_{n-1} + (-1)^n b_1 b_2 \dots b_n.$$

- 235 Write the last element of the last row in the form  $a_n - a_n$ . Establish the recursion formula

$$\Delta_n = (-1)^{n-1} b_1 b_2 \dots b_{n-1} a_n - a_n \Delta_{n-1}.$$

- 236 Subtract the last row from each of the others.



- 237 Write the first element of the first row in the form  $1 = x + (1 - x)$ . Write the determinant as the sum of two determinants and use the result of 236.
- 238, 239 Multiply the second row by  $x^{n-1}$ ; the third by  $x^{n-2}$ ; ...; the  $n$ -th by  $x$ . Factor  $x^n$  from the first column,  $x^{n-1}$  from the second column; ...;  $x$  from the  $n$ -th column.
- 240 Starting at the last column, add  $-1$  times the preceding column to each column. Then proceed similarly with the rows. Show that  $\Delta_n = \Delta_{n-1}$ . The subtractions are easily performed by noting the relation  $C_k^n = C_k^{n-1} + C_{k-1}^{n-1}$ .
- 241 Starting with the next to last, add  $-1$  times each column to the succeeding.
- 242 From the  $n$ -th, ... row subtract the preceding row; show that  $\Delta_n = \Delta_{n-1}$ .
- 243 Factor  $m$  from the first row;  $m + 1$  from the second row; ...;  $m + n$  from the last row. Factor  $\frac{1}{k}$  from the first column;  $\frac{1}{k+1}$  from the second column; ... Continue this sequence of operations until all the elements of the first column become equal to 1.
- 244 First from each column subtract the preceding. In the resulting matrix change each column except the first two by subtracting the preceding column. Continue in this way, but leaving the first three,

the first four, etc. unaltered. After this operation has been performed  $m$  times, a matrix is obtained in which all the elements of the last column are 1. It is easy to compute its determinant.

- 245 From each row, subtract the preceding and establish the recursion formula

$$\Delta_{n+1} = (x-1)\Delta_n.$$

See the hint for problem 212.

- 246 From each row subtract the preceding and establish the recursion formula

$$\Delta_{n+1} = (n-1)!(x-1)\Delta_n.$$

- 247 From each row subtract the preceding. From each column subtract the preceding. Establish the recursion formula  $\Delta_n = \alpha\Delta_{n-1}$ .

- 248 Write the last element in the last row in the form  $z + (x - z)$ . Write the determinant as the sum of two determinants. At this point make the observation that the determinant is symmetric in  $y, z$ .

- 249 See the hint for the preceding problem.

- 252 Subtract  $\beta/a$  times the first row from each of the remaining rows. Factor the quantity

$$\frac{ab - \lambda\beta}{a(\alpha - \beta)}$$

from the first column and subtract all the remaining columns from the first.

- 253, Add all the remaining columns to the first and  
 254 from each row subtract the preceding. See  
 problem 199.
- 256 Think of the determinant as a polynomial in  $a$  of the  
 fourth degree. Show that each of the following  
 linear factors divides the determinant:  
 $a + b + c + d$ ;  $a + b - c - d$ ;  $a - b + c - d$ ;  
 $a - b - c + d$ .
- 258 Add each of the last  $n - 1$  columns to the first,  
 and divide out the monomial  $x + a_1 + \cdots + a_n$ .  
 By substituting  $x = a_1, a_2, \cdots, a_n$  note that the  
 determinant is divisible by each of  
 $x - a_1, x - a_2, \cdots, x - a_n$ .
- 259 This is a Vandermonde determinant.
- 264 Expand according to minors of the elements of the  
 first column.
- 265 Subtract the first column from the second; then  
 subtract the second column from the third and so  
 on.
- 269 Factor  $\frac{1}{2!}$  from the third row;  $\frac{1}{3!}$  from the  
 fourth; ...
- 270 Use the result of problem 269.
- 271 Factor 2 from the second column; 3 from the third;  
 ... To evaluate the expression  

$$\prod_{n \geq l > k \geq 1} (l^2 - k^2)$$
  
 write it in the form

$$\prod (l^2 - k^2) = \prod (l - k) \cdot \prod (l + k).$$

- 272 Factor  $\frac{x_1}{x_1-1}$  from the first column;  
 $\frac{x_2}{x_2-1}$  from the second column; ...
- 273 Factor  $a_1^n$  from the first row;  $a_2^n$   
 from the second row; ...
- 275 To the first row add  $C_1^{2^n}$  times the second row  
 plus  $C_2^{2^n}$  times the third row, etc.
- 276 Use the result of problem 51.
- 277 Use problem 53.
- 278 Border the matrix with the row 1,  $x_1, x_2, \dots, x_n$   
 and the column 1, 0, 0, ..., 0.
- 279 Border the matrix to obtain

$$D = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ x_1 & x_2 & \dots & x_n & z \\ x_1^2 & x_2^2 & \dots & x_n^2 & z^2 \\ \dots & \dots & \dots & \dots & \dots \\ x_1^n & x_2^n & \dots & x_n^n & z^n \end{bmatrix}.$$

Find the determinant of  $D$  by expanding according  
 to minors of the last column and obtain the result

$$\det D = \prod_{n \geq i > k \geq 1} (x_i - x_k) \cdot \prod_{i=1}^n (z - x_i).$$

Find the coefficient of  $z$  in this product.

- 280 Use the hint for the preceding problem.

- 282 Border the matrix with first row  $1, 0, \dots, 0$  and first column  $1, 1, 1, \dots, 1$ . Subtract the first column from each of the others.
- 285 Expand according to minors of the elements of the last row.
- 286 First from each column starting with the last subtract  $x$  times the preceding. Then reduce the order and factor out obvious monomial factors; transform the first row (that depends on  $x$ ) by using the relation
- $$(m+1)^s - m^s = sm^{s-1} + \frac{s(s-1)}{2} m^{s-2} + \dots + 1.$$
- 287 From each column beginning with the last subtract  $x$  times the preceding.
- 288 m) To the first column add the sixth and the eleventh; to the second column the seventh and the twelfth;  $\dots$ ; to the fifth column the tenth and the fifteenth. Add the eleventh column to the sixth; the twelfth to the seventh;  $\dots$ ; the fifteenth to the tenth.

Subtract the tenth row from the fifteenth; the ninth from the fourteenth;  $\dots$ ; the first from the sixth.

- 293 Consider the product of the two matrices

$$\begin{bmatrix} 1 & C_1^n a_0 & C_2^n a_0^2 & \dots & a_0^n \\ 1 & C_1^n a_1 & C_2^n a_1^2 & \dots & a_1^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & C_1^n a_n & C_2^n a_n^2 & \dots & a_n^n \end{bmatrix} \cdot \begin{bmatrix} b_0^n & b_1^n & \dots & b_n^n \\ b_0^{n-1} & b_1^{n-1} & \dots & b_n^{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \dots & 1 \end{bmatrix}.$$

- 294 Consider the product of the two matrices

$$\begin{bmatrix} \sin \alpha_1 & \cos \alpha_1 & 0 & \dots & 0 \\ \sin \alpha_2 & \cos \alpha_2 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \sin \alpha_n & \cos \alpha_n & 0 & \dots & 0 \end{bmatrix} \cdot \begin{bmatrix} \cos \alpha_1 & \cos \alpha_2 & \dots & \cos \alpha_n \\ \sin \alpha_1 & \sin \alpha_2 & \dots & \sin \alpha_n \\ 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

- 295 Consider the product of the two matrices

$$\begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ x_1 & x_2 & \dots & x_n & x \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1^n & x_2^n & \dots & x_n^n & x^n \end{bmatrix} \cdot \begin{bmatrix} 1 & x_1 & \dots & x_1^{n-1} & 0 \\ 1 & x_2 & \dots & x_2^{n-1} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & x_n & \dots & x_n^{n-1} & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

- 296 Square the matrix.

- 297 Subtract the first column from the third; the second column from the fourth; next multiply by the matrix

$$\begin{bmatrix} \cos \varphi & -\sin \varphi & 0 & 0 \\ \sin \varphi & \cos \varphi & 0 & 0 \\ 0 & 0 & \cos 2\varphi & -\sin 2\varphi \\ 0 & 0 & \sin 2\varphi & \cos 2\varphi \end{bmatrix}.$$

- 298 Subtract  $n$  times the first column from the second;  $n$  times the second column from the fourth. Interchange the second and fourth columns. Finally multiply by the matrix

$$\begin{bmatrix} \cos n\varphi & -\sin n\varphi & 0 & 0 \\ \sin n\varphi & \cos n\varphi & 0 & 0 \\ 0 & 0 & \cos(n+1)\varphi & -\sin(n+1)\varphi \\ 0 & 0 & \sin(n+1)\varphi & \cos(n+1)\varphi \end{bmatrix}.$$

- 299 Square the matrix. Transform it as in evaluating the Vandermonde determinants; and write each difference as the sine of some angle. In this way determine the sign.
- 300 Consider the following matrix product

$$\begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \dots & a_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ a_1 & a_2 & a_3 & \dots & a_0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & \epsilon_1 & \dots & \epsilon_{n-1} \\ \dots & \dots & \dots & \dots \\ 1 & \epsilon_1^{n-1} & \dots & \epsilon_{n-1}^{n-1} \end{bmatrix},$$

where  $\epsilon_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$ .

- 308 Use the abbreviation  $\epsilon_1 = \cos \frac{\pi}{n} + i \sin \frac{\pi}{n}$ .

Then

$$\Delta = \prod_{k=0}^{n-1} \left( \sum_{j=1}^n \frac{\epsilon_1^j + \epsilon_1^{-j}}{2} \epsilon_1^{2k(j-1)} \right).$$

311 Use problem 92.

314

$$\begin{aligned} & \prod_{k=0}^{2n-1} (a_0 + a_1 \varepsilon_k + a_2 \varepsilon_k^2 + \dots + a_{2n-1} \varepsilon_k^{2n-1}) = \\ &= \prod_{r=0}^{n-1} [(a_0 + a_n) + (a_1 + a_{n+1}) \alpha_r + \dots + (a_{n-1} + a_{2n-1}) \alpha_r^{n-1}] \\ & \quad \times \prod_{s=0}^{n-1} [(a_0 - a_n) + (a_1 - a_{n+1}) \beta_s + \dots + (a_{n-1} - a_{2n-1}) \beta_s^{n-1}], \end{aligned}$$

where  $\varepsilon_k = \cos \frac{k\pi}{n} + i \sin \frac{k\pi}{n}$ ;  $\alpha_r = \cos \frac{2r\pi}{n} + i \sin \frac{2r\pi}{n}$ ;  
 $\beta_s = \cos \frac{(2s+1)\pi}{n} + i \sin \frac{(2s+1)\pi}{n}$ .

323 Subtract the first row from each of the others and the first column from each of the others.

325 Use problem 217.

327 Write the determinant as the sum of determinants or set  $x = 0$  in the determinant and its derivatives.

328 1) From each row with index  $2n - 1, 2n - 2, \dots, n + 1$ , subtract the preceding; from the  $n$ -th row subtract the sum of the others.

2) To the  $(n + i)$ -th row add the  $i$ -th,  $i = 1, 2, \dots, n - 1$ .

329 To each row add the sum of all the remaining; from each column subtract the remaining. Obtain the recursion relation  $\Delta_{n+1}(x) = (x - n)\Delta_n(x - 1)$ .



## CHAPTER IV - HINTS

## MATRICES

- 466 Use the result of problem 465 e).
- 473 Consider the sum of the elements in the principal diagonal.
- 491 Use the result of problems 489, 490.
- 492 Use the result of problem 490.
- 494, 495 Use the results of problems 492, 493.
- 496 Construct a proof by using induction on the number of columns of the matrix  $B$ . First show that if addition of one column does not change the rank of the matrix  $B$ , then it cannot change the rank the matrix  $(A, B)$ . One can also establish the theorem from the Laplace expansion without using any induction.
- 497 Use the results of problems 496, 492.
- 498 Find a non-singular square submatrix  $D$  of the matrix  $(E - A, E + A)$  and consider the products  $(E - A)P$ ,  $(E + A)P$ .
- 500 Use the result of problem 489.
- 501 Show that the representation in problem 500 is unique and reduce the problem to the enumeration of the number of triangular matrices  $R$  with given determinant  $k$ . Calling this number  $F_n(k)$ , show that if  $k = ab$ , where  $a, b$  are relatively prime,

then  $F_n(k) = F_n(a) \cdot F_n(b)$ . Finally, use induction to obtain a formula for  $F_n(p^m)$ , where  $p$  is a prime number.

- 505 Use the results of problems 495, 498. Choose a matrix  $P$  with smallest possible determinant, and so that  $P^{-1}AP$  is diagonal. Then use the result of problem 500.
- 517 Use Buniakovsky's inequality in Laplace's expansion.
- 518 If the sum of the products of the elements of an arbitrary column of  $B$  by the corresponding elements of an arbitrary column of  $C$  is 0, show that the equality  $\det A'A = (\det B'B)(\det C'C)$  holds. Then border the matrix  $(D, C)$  so as to obtain a square matrix and use the result of problem 517.
- 523 Border the matrix with a column on the left all elements of which are  $M/2$ , and a row of  $n$  zeros. Then add the first column to each of the others.
- 527 Use the results of problems 522, 526.
- 528 Use the relation between the given matrix and its inverse.
- 529 Let  $A_{ik}$  be the algebraic cofactor of the element  $a_{ik}$ . To establish the result for the elements for the minor formed on the first  $m$  rows and the first  $m$  columns, consider the matrix product:

$$\begin{pmatrix} A_{11} & \dots & A_{m+1,1} & \dots & A_{n1} \\ A_{12} & \dots & A_{m+1,2} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots & \dots \\ A_{1m} & \dots & A_{m+1,m} & \dots & A_{nm} \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix},$$

The general result is obtained similarly.

535 Write  $A \times B$  in the form  $(A \times E_m) \cdot (E_n \times B)$ .

537 Use an inductive proof starting with the case:

$A_{11}$  is a nonsingular matrix. The general case can be established by adding  $\lambda E$  if necessary.

CHAPTER V - HINTS  
POLYNOMIALS AND RATIONAL FUNCTIONS  
IN A SINGLE INDETERMINATE

- 547 a) First expand  $f(x)$  in powers of  $x - 3$ ; then replace  $x$  by  $x + 3$ .
- 553 In the derivative first set  $x = 1$ ; then factor out as high a power of  $x$  as possible and differentiate again.
- 555 Introduce the sequence of auxiliary polynomials  

$$f_1(x) = nf(x) - xf'(x); \quad f_2(x) = nf_1(x) - xf_1'(x), \dots$$
- 561 Establish the result by mathematical induction.
- 562 A non-zero  $(k - 1)$ -fold root of the polynomial  $f(x)$  is a  $(k - 2)$ -fold root of the polynomial  $xf'(x)$ ; a  $(k - 3)$ -fold root of the polynomial  $x[x f'(x)]'$ ; ...
- Conversely any nonzero root common to the polynomials  $f(x)$ ,  $xf'(x)$ ,  $x[x f'(x)]'$  (a sequence of  $k - 1$  polynomials) is a root of  $f(x)$  of multiplicity at least  $k - 1$ .
- 563 Divide the polynomial by its derivative and differentiate the resulting equality.
- 567 Consider one of the functions  $\frac{f_1(x)}{f_2(x)}$ ,  $\frac{f_2(x)}{f_1(x)}$ .
- 568 Let  $x_0$  be a root of  $[f'(x)]^2 - f(x) f''(x)$ . Then examine the roots of  $\varphi(x) = f(x) f'(x_0) - f'(x) f(x_0)$ .
- 569 Expand  $f(x)$  in powers of  $x - x_0$ ; use the result of the preceding problem.

576 The proof is similar to the proof of D'Alembert's lemma.

580, Write the function as follows (the form used in the  
581 proof of D'Alembert's lemma):

$$f(z) = f(a) + \frac{f^{(k)}(a)}{k!} (z-a)^k [1 + \varphi(z)]; \quad \varphi(a) = 0.$$

583 In a) and b) find the roots of the polynomial and factor the coefficient of the highest power of  $x$ . In c) the roots are conveniently found by making the substitution  $x = \operatorname{tg}^2 \theta$ .

589 Find the common roots.

608 As a first step show that  $f(x)$  has no real roots of odd multiplicity.

623 Use the result of problem 622.

626 Note that the equation is unchanged if  $x$  is replaced by  $-x$  or by  $1/x$ .

627 Note that the equation is unchanged if  $x$  is replaced by  $1/x$  or by  $1 - x$ .

637 Factor out  $(1 - x)^n$ . Differentiate  $m - 1$  times. Set  $x = 0$  in the result of each differentiation. Use the fact that the degree of  $N(x)$  is less than  $m$ ; the degree of  $M(x)$  is less than  $n$ .

642 Use Lagrange's formula. Perform the division in every term of the result and use the result of Problem 100 to simplify the individual terms.

- 644 First use Lagrange's interpolation formula to express  $f(x_0)$  in terms of  $f(x_1), f(x_2), \dots, f(x_n)$ . Substitute this in the given equation. Note that the quantities  $f(x_1), f(x_2), \dots, f(x_n)$  are independent. Finally use the formula  $\varphi(x) = (x - x_1)(x - x_2) \dots (x - x_n)$ , and expand in powers of  $x - x_0$ .
- 645 The polynomial  $x^s$  is determined by its values, by means of Lagrange's interpolation formula.
- 648, 649 Use Newton's interpolation formula.
- 650 Find the values of the polynomial for  $x = 0, 1, 2, 3, \dots, 2n$ .
- 651 The problem can be solved by using Newton's interpolation formula. Another way is to consider the polynomial  $F(x) = x f(x) - 1$ , where  $f(x)$  is the given polynomial.
- 652 Consider the polynomial  $(x - a) f(x) - 1$ .
- 653 Use Newton's interpolation formula. It is convenient to introduce the factorial in the denominator of every term.
- 654 Let  $f(x)$  be the given polynomial; consider the polynomial  $f(x^2)$ .

- 655 If  $x_1, x_2, \dots, x_n$  are the roots of the denominator, then Lagrange's formula reads:

$$\frac{f(x)}{\varphi(x)} = \sum_{k=1}^n \frac{f(x_k)}{(x-x_k)\varphi'(x_k)}$$

- 656 First use the Lagrange methods explained in the Hint above; then select terms that are complex conjugates.

- 657 e) Use Problem 631. f) Set  $\frac{a+x}{2a} = y$ .

d), h) Write out the general form of the answer. Multiply through by the greatest common denominator and set  $x = x_1, x_2, \dots, x_n$ . Then make the same substitution after differentiating the identity.

- 660 Use Problem 659. For part b) separate  $\frac{1}{x^2-3x+2}$  into partial fractions.

- 665, Use the result of Problem 663.  
666

- 667 In part c) expand the polynomial in powers of  $x-1$ .

- 668 Expand in the powers of  $x-1$  [set  $x = y+1$ ].

- 669 Set  $x = y+1$ ; use mathematical induction to prove that all the coefficients of the dividend and divisor (except the coefficient of the highest power) are divisible by  $p$ .

- 670, The proof is similar to the proof of Eisenstein's  
671 criterion.

- 679, Assuming that  $f(x)$  is reducible, set  
680  $x = a_1, a_2, \dots, a_n$  and note what values the factors of  $f(x)$  would have to have.
- 681 If proper divisors existed, find how many equal values could occur.
- 682 Use the fact that  $f(x)$  has no real roots.
- 683 Show that a polynomial that has more than three integral roots cannot take a prime value for an integral value of its argument. Apply this remark to the polynomial  $f(x) - 1$ .
- 684, Use the result of Problem 683.  
685
- 702 Set up a Sturm sequence and consider the cases  $n$  odd,  $n$  even separately.
- 707- Find recurrence relations among the polynomials of  
712 consecutive degrees and their derivatives; construct Sturm sequences from these. In Problem 708 set up a Sturm sequence only for positive values of  $x$  and use other arguments to prove the non-existence of negative roots. In Problem 709 set up a Sturm sequence for negative  $x$ .
- 713 Use the fact that  $F'(x) = 2f(x) f'''(x)$ , and that  $f'''(x)$  is constant.
- 717 Use the factored form of  $g(x)$ ; use the results of 716 several times.
- 718 Apply 717 in the special case  $g(x) = x^m$ .



719 Use the fact that if all roots of the polynomial

$$a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n \quad \text{are real,}$$

then all roots of the polynomial

$$a_nx^n + a_{n-1}x^{n-1} + \dots + a_0 \quad \text{are also real.}$$

721 Multiply by  $x - 1$ .

727 Assume the assertion false; use Rolle's theorem and the result of Problem 581.

728 Consider the graph of  $\psi(x) = \frac{f(x)}{f'(x)}$ . Show that every root of  $[f'(x)]^2 - f(x)f''(x)$  corresponds to an extremum of  $\psi(x)$  and conversely. Show that  $\psi(x)$  cannot have extrema in an interval bounded by roots of  $f'(x)$  that contains a root of  $f(x)$ ; and can have only one extremum in an interval that does not contain a root of  $f(x)$ .

729 Use the results of Problems 727, 726.

730 Elucidate the behavior of the function

$$\psi(x) = \frac{f(x)}{f'(x)} + \frac{x+\lambda}{\gamma}.$$

731 Set  $\lambda = 0$  in the result of the preceding problem.

732 Use induction on the degree of  $f(x)$ ; set

$$f(x) = (x + \lambda) f_1(x), \text{ where } f_1(x) \text{ is a polynomial of degree } n - 1.$$

733 Apply the result of preceding problem twice.

734 If all the roots of  $f(x)$  are positive, then the proof follows from elementary considerations using induction on the degree of  $f(x)$ . As an

inductive hypothesis use the statement that the roots  $x_1, x_2, \dots, x_{n-1}$  of the polynomial

$b_0 + b_1 \omega x + \dots + b_{n-1} \omega^{(n-1)^2} x^{n-1}$  satisfy the relations

$$0 < x_1 < x_2 < \dots < x_{n-1}, \quad x_i > x_{i-1} \omega^{-2}.$$

To prove the theorem in the general case write  $\omega^{x^2}$  as the limit of a polynomial in  $x$  with the roots lying exterior to the interval  $(0, n)$ ; and use the result of Problem 731.

735, Consider  $\left| \frac{\varphi(x) + i\psi(x)}{\varphi(x) - i\psi(x)} \right|$ , where  
736

$$\begin{aligned}\varphi(x) &= a_0 \cos \varphi + \dots + a_n \cos(\varphi + n\theta) x^n, \\ \psi(x) &= b_0 \sin \varphi + \dots + b_n \sin(\varphi + n\theta) x^n.\end{aligned}$$

or

$$\begin{aligned}\varphi(x) &= a_0 + a_1 x + \dots + a_n x^n, \\ \psi(x) &= b_0 + b_1 x + \dots + b_n x^n.\end{aligned}$$

In 736, to show the roots are real, multiply  $\varphi(x) + i\psi(x)$  by  $\alpha - \beta i$  and examine the real part. Use the result of Problem 727.

737 Resolve  $\frac{\psi(x)}{\varphi(x)}$  into partial fractions. Note the signs of the coefficients in this resolution, and examine the imaginary part of

$$\frac{-i[\varphi(x) + i\psi(x)]}{\varphi(x)} = \frac{\psi(x)}{\varphi(x)} - i.$$

- 738 Examine the imaginary part of  $\frac{f'(x)}{f(x)}$ , when it is written as a sum of partial fractions.
- 739 Note that a simple substitution of the argument will transform the given half plane into the half plane  $\text{Im}(x) > 0$ ; see the preceding problem.
- 740 Show how to apply Problem 739.
- 741 Separate  $\frac{f'(x)}{f(x)}$  into partial fractions and evaluate the imaginary parts.
- 743 Set  $x = yi$  and use the result of Problems 736, 737.
- 744, 745 Use the result of Problem 743.
- 746 Set  $x = (1 + y)/(1 - y)$  and use the result of Problem 744.
- 747 Multiply the polynomial by  $1 - x$  and calculate the modulus of  $(1 - x) f(x)$  for  $|x| = \rho > 1$ .

CHAPTER VI - HINTS  
SYMMETRIC FUNCTIONS

772 If a triangle similar to the given triangle is inscribed in a circle of radius  $1/2$ , its sides will be equal to the sines of the angles of the given triangle.

800, First consider the sum  
801

$$\sum_{i=1}^n (x + x_i)^k,$$

then set  $x = x_j$  and sum on  $j$  from 1 to  $n$ . Finally, combine repeated terms and divide by 2.

805 Every primitive  $n$ -th root of unity when raised to the  $m$ -th power gives a primitive  $(n/d)$ -th root, where  $d = (m, n)$  is the greatest common divisor of  $m$  and  $n$ . Moreover if we operate in this way on the primitive  $n$ -th roots of unity, we obtain all the primitive  $(n/d)$ -th roots of unity without repetition.

806 Use the results of problems 805, 117, 119.

807 Note that the equation that has roots  $x_1, x_2, \dots, x_n$  is the determinantal equation of problem 803; this follows from Newton's formula expressing the coefficients in terms of power sums of the roots.

- 808 The problem is easily solved by using Newton's formula or by expressing the Newton elementary functions in terms of the fundamental symmetric functions. See the determinant in problem 802. Another method is to multiply the equation by  $(x - a) \cdot (x - b)$  and calculate the power sums in the resulting equation.
- 809 First multiply the equation by  $(x - a) \cdot (x - b)$ .
- 818 Think of the roots of the polynomial  $f(x)$  as independent variables. Multiply the determinant of coefficients by the Vandermondian.
- 819, 820 First show that every polynomial  $\psi_k$  has degree  $n - 1$ . Then multiply the determinant of the coefficients of  $\psi_k$  by the Vandermondian.
- 827 Use the fact that the primitive  $m$ -th roots of the  $n$ -th roots of unity run through all the primitive  $(n/d)$ -th roots of unity, where  $d = (m, n)$ .
- 828 Use the result of problem 827 and the fact that  $R(X_m, X_n)$  is a divisor of  $R(X_m, x^n - 1)$  and  $R(X_n, x^m - 1)$ .
- 834, 835 Calculate  $R(f', f)$ .
- 839 Multiply by  $x - 1$ .
- 840 Multiply by  $x - 1$  and use the result of Problem 835.

- 843 Calculate  $R(X_n, X'_n)$ . To calculate  $X'_n$  at the roots of  $X_n$ , write  $X_n$  in the form

$$(x^n - 1) \prod (x^d - 1)^{\mu \left(\frac{n}{d}\right)},$$

where  $d$  runs through the proper divisors of  $n$ .

- 844 Use the relation  $E'_n = E_n - x^n$ .

- 845 Use the relation

$$(nx - x - a)F_n - x(x+1)F'_n + \frac{(a-1) \dots (a-n)}{n!} = 0.$$

- 846 Use the relation

$$P_n = xP_{n-1} - (n-1)P_{n-2}; \quad P'_n = nP_{n-1}.$$

- 847 Use the relation

$$xP'_n = nP_n + n^2P_{n-1}; \quad P_n = (x - 2n + 1)P_{n-1} - (n-1)^2P_{n-2}.$$

- 848 Use the relation

$$(4 - x^2)P'_n + nxP_n = 2nP_{n-1}; \quad P_n - xP_{n-1} + P_{n-2} = 0.$$

- 849 Use the relation

$$P_n - 2xP_{n-1} + (x^2 + 1)P_{n-2} = 0; \quad P'_n = (n+1)P_{n-1}.$$

- 850 Use the relation

$$P_n - (2n-1)xP_{n-1} + (n-1)^2(x^2 + 1)P_{n-2} = 0; \quad P'_n = n^2P_{n-1}.$$

- 851 Use the relation

$$P_n - (2nx + 1)P_{n-1} + n(n-1)x^2P_{n-2} = 0; \quad P'_n = (n+1)nP_{n-1}.$$

- 852 Solve the problem by using the Lagrange polynomial. Note that the maximum in question can be obtained by solving a differential equation; namely setting the derivative equal to 0. Solve this equation by the method of undetermined coefficients.
- 867 First show that for fixed  $n$ , there is only a finite number of equations with the given properties. Then show that the given property is invariant under the substitution  $y = x^m$ .

## CHAPTER VII - HINTS

## LINEAR ALGEBRA

- 884 Use the results of problems 51, 52.
- 916 The smallest angle must be one of the angles formed by the vectors of the second plane with its orthogonal projection on the first plane.

- 917 Let the center of the cube be the origin and let its sides be parallel to the coordinate axes. Next take four mutually orthogonal diagonals as axes.
- 918 Use the result of problem 907.
- 920 Use induction.
- 921 Use the fact that  
 $V[A_1, \dots, A_m, B_1, \dots, B_k] = V[A_1, \dots, A_m] \cdot V[B_1, \dots, B_k]$ ,  
 if  $A_i \perp B_j$ . Then use the result of the preceding problem.
- 933 First find the proper values of the square matrix. Then to determine the sign of the square roots use the fact that the sum of the proper values is equal to the sum of the elements on the principal diagonal and that the product of the proper values is equal to the value of the determinant. Use the results of problems 126, 299.
- 934 Use the result of problem 933.
- 936 Use the results of problems 537, 930.



943 1) Use the fact that the determinant of a triangular transformation is unity.

2) Set  $x_{k+1} = x_{k+2} = \dots = x_n = 0$ .

945 Take as a new independent variable the linear form of the problem.

946 Use the result of problem 945, by writing  $f$  as the sum of the square of a linear form and other terms.

948 Let  $x_1, x_2, \dots, x_n$  be the roots of the given equation. Consider the following quadratic form in the variables  $u_1, u_2, \dots, u_n$ :

$$f = \sum_{k=1}^n (u_1 + u_2 x_k + \dots + u_n x_k^{n-1})^2.$$

950 Expand  $f$ ,  $\varphi$  as a sum of squares and use the distributivity of the operation  $(f, \varphi)$ .

965 The converse of the theorem is proved by using the relation  $X = AX + (E - A) X$ .

966 Write the matrix of the projection in terms of a basis that is obtained by complementing orthonormal bases of  $P, Q$ .

967 Note that  $(AX, X) = 0$  for an arbitrary real vector  $X$ . Write a proper value and the corresponding vector as the sum of real and imaginary parts.

968 Let  $P$  be an orthogonal matrix, the first two columns of which consist of the normed real and

imaginary parts of a proper vector; form the product  $P^{-1}AP$ .

970, Use the fact that if  $X, Y$  are arbitrary real  
971 vectors, and  $A$  is an arbitrary orthogonal matrix, then the relation  $(AX, AY) = (A'AX, Y) = (X, Y)$  holds.

972 The proof is based on the results of problems 970, 971 in the same way that the proof of 968 is based on the result of problem 967.

975, Transform to Jordan canonical form.  
976

978 This problem is related to the solution of the preceding problem.

980 For the necessity, see problem 473. To establish the sufficiency first consider the case when all the diagonal elements of the matrix  $C$  are zero. Then note that if  $C = XY - YX$ , then the relation  

$$S^{-1}CS = (S^{-1}XS)(S^{-1}YS) - (S^{-1}YS)(S^{-1}XS)$$
 holds.

981 Try  $H = \sum C^k C^{*k}$ .

982 Try  $H = \sum C^k BC^{*k}$ .

983 The matrix  $C = (I + A)(I - A)^{-1}$  has all its proper values in the unit circle.



# CHAPTER I - SOLUTIONS

## COMPLEX NUMBERS

- 1  $x = -4/11$ ;  $y = 5/11$ .
- 2  $x = -2$ ;  $y = 3/2$ ;  $z = 2$ ;  $t = -1/2$ .
- 3 1, if  $n = 4k$ ; i, if  $n = 4k + 1$ ; -1, if  $n = 4k + 2$ ;  
-1, if  $n = 4k + 3$ ;  $k$  being an integer.
- 5 a)  $117 + 44i$ ; b)  $-556$ ; c)  $-76i$ .
- 6 Only in the following cases:
  - 1) If one of the factors is not zero;
  - 2) If the factors have the form  $(a + bi)$  or  $\lambda(b + ai)$ , where  $\lambda$  is real.
- 7 a)  $\cos 2\alpha + i \sin 2\alpha$ ; b)  $\frac{a^2 - b^2}{a^2 + b^2} + i \frac{2ab}{a^2 + b^2}$ ; c)  $\frac{44 - 5i}{318}$ ;  
d)  $\frac{-1 - 32i}{25}$ ; e) 2.
- 8  $2i^{n-1}$ .
- 9 a)  $x = 1 + i$ ,  $y = i$ ; b)  $x = 2 + i$ ,  $y = 2 - i$ ;  
c)  $x = 3 - 11i$ ,  $y = -3 - 9i$ ,  $z = 1 - 7i$ .
- 10 a)  $-\frac{1}{2} - i \frac{\sqrt{3}}{2}$ ; b) 1.
- 11 a)  $a^2 + b^2 + c^2 - (ab + bc + ac)$ ; b)  $a^3 + b^3$ ;  
c)  $2(a^3 + b^3 + c^3) - 3(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b) + 12abc$ ;  
d)  $a^2 - ab + b^2$ .

- 12 a)  $0, 1, -\frac{1}{2} + \frac{i\sqrt{3}}{2}, -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ ; b)  $0, 1, i, -1, -i$ .
- 15 a)  $\pm(1+i)$ ; b)  $\pm(2-2i)$ ; c)  $\pm(2-i)$ ; d)  $\pm(1+4i)$ ;  
 e)  $\pm(1-2i)$ ; f)  $\pm(5+6i)$ ; g)  $\pm(1+3i)$ ; h)  $\pm(1-3i)$ ;  
 i)  $\pm(3-i)$ ; j)  $\pm(3+i)$ ; k)  $\pm\left(\sqrt{\frac{\sqrt{13}+2}{2}} - i\sqrt{\frac{\sqrt{13}-2}{2}}\right)$ ;  
 l)  $\pm\sqrt{8+2\sqrt{17}}$ ;  $\pm i\sqrt{-8+2\sqrt{17}}$ ; m)  $\pm\left(\sqrt{\frac{3}{2}} - i\sqrt{\frac{1}{2}}\right)$ ;  
 n)  $\frac{\sqrt{2}(\pm 1 \pm i)}{2}$ ; o)  $i^\alpha\left(\frac{1+\sqrt{3}}{2} + \frac{1-\sqrt{3}}{2}i\right)$ ,  $\alpha=0, 1, 2, 3$ .
- 16  $\pm(\beta - \alpha i)$ .
- 17 a)  $x_1 = 3 - i$ ;  $x_2 = -1 + 2i$ ; b)  $x_1 = 2 + i$ ;  $x_2 = 1 - 3i$ ;  
 c)  $x_1 = 1 - i$ ;  $x_2 = \frac{4-2i}{5}$ .
- 18 a)  $1 \pm 2i$ ;  $-4 \pm 2i$ ;  $(x^2 - 2x + 5)(x^2 + 8x + 20)$ ;  
 b)  $2 \pm i\sqrt{2}$ ;  $-2 \pm 2i\sqrt{2}$ ;  $(x^2 - 4x + 6)(x^2 + 4x + 12)$ .
- 19 a)  $x = \pm \frac{\sqrt{7}}{2} \pm \frac{i}{2}$ ; b)  $\pm 4 \pm i$ .
- 20  $\pm\sqrt{\frac{\sqrt{q}}{2} - \frac{p}{4}} \pm i\sqrt{\frac{\sqrt{q}}{2} + \frac{p}{4}}$ .
- 22 a)  $\cos 0 + i \sin 0$ ; b)  $\cos \pi + i \sin \pi$ ; c)  $\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$ ;  
 d)  $\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}$ ; e)  $\sqrt{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$ ;  
 f)  $\sqrt{2}\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right)$ ; g)  $\sqrt{2}\left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}\right)$ ;  
 h)  $\sqrt{2}\left(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}\right)$ ; i)  $2\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$ ;  
 j)  $2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)$ ; k)  $2\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right)$ ;  
 l)  $2\left(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}\right)$ ; m)  $2\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)$ ;  
 n)  $3(\cos \pi + i \sin \pi)$ ; o)  $2\left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right)$ ;  
 p)  $(\sqrt{2} + \sqrt{6})\left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12}\right)$ .

Remark: Here we have given just one of the possible values of the argument.

- 23      a)  $\sqrt{10}(\cos 18^\circ 26' + i \sin 18^\circ 26')$ ;  
         b)  $\sqrt{17}(\cos 345^\circ 57' 48'' + i \sin 345^\circ 57' 48'')$ ;  
         c)  $\sqrt{5}(\cos 153^\circ 26' 6'' + i \sin 153^\circ 26' 6'')$ ;  
         d)  $\sqrt{5}(\cos 243^\circ 26' 6'' + i \sin 243^\circ 26' 6'')$ .
- 24      a) A circle of radius 1 with center at the origin.  
         b) A half ray from the origin making an angle  $\pi/6$  with the polar axis.
- 25      a) The interior of a circle with center at the origin and radius 2.  
         b) The interior and circumference of a circle with center at the point (0, 1) and radius 1.  
         c) The interior of a circle with center at the point (1, 1) and radius 1.
- 26      a)  $x = 3/2 - 2i$ ;    b)  $x = 3/4 + i$ .
- 27      The result is equivalent to the following geometric proposition: The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides.
- 29      When the difference between the arguments of these numbers is  $\pi + 2k\pi$ , for some integer  $k$ .
- 30      When the difference in the arguments of these numbers is an integral multiple of  $2\pi$ .

$$34 \quad \cos(\varphi + \psi) + i \sin(\varphi + \psi).$$

$$35 \quad \frac{\sqrt{2}}{2} \left[ \cos\left(2\varphi - \frac{\pi}{12}\right) + i \sin\left(2\varphi - \frac{\pi}{12}\right) \right].$$

$$36 \quad \text{a) } 2^{12}(1+i); \text{ b) } 2^9(1-i\sqrt{3}); \text{ c) } (2-\sqrt{3})^{12}; \text{ d) } -64.$$

$$38 \quad \cos \frac{n\pi}{3} + i \sin \frac{n\pi}{3}.$$

$$39 \quad 2 \cos \frac{2n\pi}{3}.$$

$$\begin{aligned} 40 \quad \text{Solution.} \quad 1 + \cos \alpha + i \sin \alpha &= \\ &= 2 \cos^2 \frac{\alpha}{2} + 2i \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} = 2 \cos \frac{\alpha}{2} \left( \cos \frac{\alpha}{2} + i \sin \frac{\alpha}{2} \right); \\ (1 + \cos \alpha + i \sin \alpha)^n &= 2^n \cos^n \frac{\alpha}{2} \left( \cos \frac{n\alpha}{2} + i \sin \frac{n\alpha}{2} \right). \end{aligned}$$

$$\begin{aligned} 43 \quad \text{a) } &-i; \frac{\sqrt{3}+i}{2}; \frac{-\sqrt{3}+i}{2}; \\ \text{b) } &-1+i; \frac{1+\sqrt{3}}{2} + \frac{\sqrt{3}-1}{2}i; \frac{1-\sqrt{3}}{2} - \frac{1+\sqrt{3}}{2}i; \\ \text{c) } &1+i; 1-i; -1+i; -1-i; \\ \text{d) } &1; -1; -\frac{1}{2} + i\frac{\sqrt{3}}{2}; -\frac{1}{2} - i\frac{\sqrt{3}}{2}; \frac{1}{2} + i\frac{\sqrt{3}}{2}; \frac{1}{2} - i\frac{\sqrt{3}}{2}; \\ \text{e) } &i\sqrt{3}; -i\sqrt{3}; \frac{3+i\sqrt{3}}{2}; \frac{3-i\sqrt{3}}{2}; -\frac{3+i\sqrt{3}}{2}; -\frac{3-i\sqrt{3}}{2}. \end{aligned}$$

$$44 \quad \text{a) } \sqrt[6]{5}(\cos 8^\circ 5' 18'' + i \sin 8^\circ 5' 18'') \varepsilon_k,$$

$$\text{where } \varepsilon_k = \cos 120^\circ k + i \sin 120^\circ k, \quad k = 0, 1, 2;$$

$$\text{b) } \sqrt[6]{10}(\cos 113^\circ 51' 20'' + i \sin 113^\circ 51' 20'') \varepsilon_k,$$

$$\text{where } \varepsilon_k = \cos 120^\circ k + i \sin 120^\circ k, \quad k = 0, 1, 2;$$

$$\text{c) } \sqrt[10]{13}(\cos 11^\circ 15' 29'' + i \sin 11^\circ 15' 29'') \varepsilon_k,$$

$$\text{where } \varepsilon_k = \cos 72^\circ k + i \sin 72^\circ k, \quad k = 0, 1, 2, 3, 4.$$

$$45 \quad a) \quad \frac{1}{\sqrt[12]{2}} \left( \cos \frac{24k+19}{72} \pi + i \sin \frac{24k+19}{72} \pi \right),$$

where  $k = 0, 1, 2, 3, 4, 5$ ;

$$b) \quad \frac{1}{\sqrt[16]{2}} \left( \cos \frac{24k+5}{96} \pi + i \sin \frac{24k+5}{96} \pi \right),$$

where  $k = 0, 1, 2, 3, 4, 5, 6, 7$ ;

$$c) \quad \frac{1}{\sqrt[12]{2}} \left( \cos \frac{24k+17}{72} \pi + i \sin \frac{24k+17}{72} \pi \right),$$

for  $k = 0, 1, 2, 3, 4, 5$ .

$$46 \quad \beta \left( \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} \right), \quad \text{for } k = 0, 1, 2, \dots, n-1.$$

47 a) Solution. Consider  $(\cos x + i \sin x)^5$ . By DeMoivre's theorem, we have

$$(\cos x + i \sin x)^5 = \cos 5x + i \sin 5x.$$

On the other hand,

$$\begin{aligned} (\cos x + i \sin x)^5 &= \cos^5 x + 5i \cos^4 x \sin x - 10 \cos^3 x \sin^2 x - \\ &\quad - 10i \cos^2 x \sin^3 x + 5 \cos x \sin^4 x + i \sin^5 x = \\ &= (\cos^5 x - 10 \cos^3 x \sin^2 x + 5 \cos x \sin^4 x) + i(5 \cos^4 x \sin x - \\ &\quad - 10 \cos^2 x \sin^3 x + \sin^5 x). \end{aligned}$$

Collecting these results we obtain:

$$\cos 5x = \cos^5 x - 10 \cos^3 x \sin^2 x + 5 \cos x \sin^4 x;$$

$$b) \quad \cos^5 x - 28 \cos^6 x \sin^2 x + 70 \cos^4 x \sin^4 x - 28 \cos^2 x \sin^6 x + \sin^8 x;$$

$$c) \quad 6 \cos^5 x \sin x - 20 \cos^3 x \sin^3 x + 6 \cos x \sin^5 x;$$

$$d) \quad 7 \cos^6 x \sin x - 35 \cos^4 x \sin^3 x + 21 \cos^2 x \sin^5 x - \sin^7 x.$$

$$48 \quad \frac{2(3 \operatorname{tg} \varphi - 10 \operatorname{tg}^3 \varphi + 3 \operatorname{tg}^5 \varphi)}{1 - 15 \operatorname{tg}^2 \varphi + 15 \operatorname{tg}^4 \varphi - \operatorname{tg}^6 \varphi}.$$



$$49 \quad \cos nx = \cos^n x - C_2^n \cos^{n-2} x \sin^2 x + C_4^n \cos^{n-4} x \sin^4 x - \dots + M,$$

where  $M = (-1)^{\frac{n}{2}} \sin^n x$ , if  $n$  is even;

$$M = (-1)^{\frac{n-1}{2}} n \cos x \sin^{n-1} x, \quad \text{if } n \text{ is odd.}$$

$$\sin nx = C_1^n \cos^{n-1} x \sin x - C_3^n \cos^{n-3} x \sin^3 x + \dots + M,$$

where  $M = (-1)^{\frac{n-2}{2}} n \cos x \sin^{n-1} x$ ,  $n$  is even;

$$M = (-1)^{\frac{n-1}{2}} \sin^n x, \quad \text{if } n \text{ is odd.}$$

50 a) Solution. Let  $\alpha = \cos x + i \sin x$ . Then

$$\alpha^{-1} = \cos x - i \sin x;$$

$$\alpha^k = \cos kx + i \sin kx; \quad \alpha^{-k} = \cos kx - i \sin kx.$$

$$\text{Thus we obtain } \cos kx = \frac{\alpha^k + \alpha^{-k}}{2}; \quad \sin kx = \frac{\alpha^k - \alpha^{-k}}{2i}.$$

In particular

$$\cos x = \frac{\alpha + \alpha^{-1}}{2}; \quad \sin x = \frac{\alpha - \alpha^{-1}}{2i};$$

$$\sin^3 x = \left( \frac{\alpha - \alpha^{-1}}{2i} \right)^3 = \frac{\alpha^3 - 3\alpha + 3\alpha^{-1} - \alpha^{-3}}{-8i} = \frac{(\alpha^3 - \alpha^{-3}) - 3(\alpha - \alpha^{-1})}{-8i};$$

$$\sin^3 x = \frac{2i \sin 3x - 6i \sin x}{-8i} = \frac{3 \sin x - \sin 3x}{4};$$

$$\text{b) } \frac{\cos 4x - 4 \cos 2x + 3}{8}; \quad \text{c) } \frac{\cos 5x + 5 \cos 3x + 10 \cos x}{16};$$

$$\text{d) } \frac{\cos 6x + 6 \cos 4x + 15 \cos 2x + 10}{32}.$$

52 Solution.

$$C_p^{m-p} + C_{p-1}^{m-p-1} = \frac{(m-p)(m-p-1) \dots (m-2p+1)}{p!} +$$

$$+ \frac{(m-p-1) \dots (m-2p+1)}{(p-1)!} = \frac{m(m-p-1)(m-p-2) \dots (m-2p+1)}{p!}.$$

Set  $2 \cos mx = S_m$ ;  $2 \cos x = a$ . Then the equality we are looking for can be written in the following form:

$$S_m = a^m - ma^{m-2} + (C_2^{m-2} + C_1^{m-3})a^{m-4} - \dots \\ \dots + (-1)^p (C_p^{m-p} + C_{p-1}^{m-p-1})a^{m-2p} + \dots$$

It is not difficult to show that:

$$2 \cos mx = 2 \cos x \cdot 2 \cos (m-1)x - 2 \cos (m-2)x,$$

$$\text{or in our notation } S_m = aS_{m-1} - S_{m-2}.$$

We first check that for  $m=1$ ,  $m=2$  the inequality we are trying to prove is valid. Then we use induction. We assume that

$$S_{m-1} = a^{m-1} - (m-1)a^{m-3} + (C_2^{m-3} + C_1^{m-4})a^{m-5} + \dots \\ \dots + (-1)^p (C_p^{m-p-1} + C_{p-1}^{m-p-2})a^{m-2p-1} + \dots; \\ S_{m-2} = a^{m-2} - (m-2)a^{m-4} + (C_2^{m-4} + C_1^{m-5})a^{m-6} + \dots \\ \dots + (-1)^{p-1} (C_{p-1}^{m-p-1} + C_{p-2}^{m-p-2})a^{m-2p} + \dots$$

$$\text{Then } S_m = a^m - ma^{m-2} + \dots \\ \dots + (-1)^p (C_p^{m-p-1} + C_{p-1}^{m-p-2} + C_{p-1}^{m-p-1} + C_{p-2}^{m-p-2})a^{m-2p} + \dots$$

Using the fact that  $C_k^n = C_k^{n-1} + C_{k-1}^{n-1}$ , we establish the desired result.

$$53 \quad \frac{\sin mx}{\sin x} = (2 \cos x)^{m-1} - C_1^{m-2} (2 \cos x)^{m-3} + \\ + C_2^{m-3} (2 \cos x)^{m-5} - \dots + (-1)^p C_p^{m-p-1} (2 \cos x)^{m-2p-1} + \dots$$

54 a)  $2^{\frac{n}{2}} \cos \frac{n\pi}{4}$ ; b)  $2^{\frac{n}{2}} \sin \frac{n\pi}{4}$ .

56  $\frac{2^n}{3^{\frac{n-1}{2}}} \sin \frac{n\pi}{6}$ .

59 a) Solution.

$$S = 1 + a \cos \varphi + a^2 \cos 2\varphi + \dots + a^k \cos k\varphi.$$

We find  $T = a \sin \varphi + a^2 \sin 2\varphi + \dots + a^k \sin k\varphi$ ;

$$S + Ti = 1 + a(\cos \varphi + i \sin \varphi) + a^2(\cos 2\varphi + i \sin 2\varphi) + \dots \\ \dots + a^k(\cos k\varphi + i \sin k\varphi)$$

Set  $\alpha = \cos \varphi + i \sin \varphi$ . Then

$$S + Ti = 1 + a\alpha + a^2\alpha^2 + \dots + a^k\alpha^k = \frac{a^{k+1}\alpha^{k+1} - 1}{a\alpha - 1}.$$

S is the real part of this sum. Moreover,

$$S + Ti = \frac{a^{k+1}\alpha^{k+1} - 1}{a\alpha - 1} \cdot \frac{a\alpha^{-1} - 1}{a\alpha^{-1} - 1} = \frac{a^{k+2}\alpha^k - a^{k+1}\alpha^{k+1} - a\alpha^{-1} + 1}{a^2 - a(\alpha + \alpha^{-1}) + 1}.$$

Thus  $S = \frac{a^{k+2} \cos k\varphi - a^{k+1} \cos (k+1)\varphi - a \cos \varphi + 1}{a^2 - 2a \cos \varphi + 1}.$

b)  $\frac{a^{k+2} \sin (\varphi + kh) - a^{k+1} \sin [\varphi + (k+1)h] - a \sin (\varphi - h) + \sin \varphi}{a^2 - 2a \cos h + 1}.$

c)  $\frac{\sin \frac{2n+1}{2} x}{2 \sin \frac{x}{2}}.$

60 Solution.

$$T = \sin x + \sin 2x + \dots + \sin nx;$$

$$S = \cos x + \cos 2x + \dots + \cos nx.$$

Let  $\alpha = \cos \frac{x}{2} + i \sin \frac{x}{2}$  Then  $S + Ti = \alpha^2 + \alpha^4 + \dots + \alpha^{2n}$ ,

$$S + Ti = \alpha^2 \frac{\alpha^{2n} - 1}{\alpha^2 - 1} = \alpha^2 \frac{\alpha^n (\alpha^n - \alpha^{-n})}{\alpha (\alpha - \alpha^{-1})} = \left( \cos \frac{n+1}{2} x + i \sin \frac{n+1}{2} x \right) \frac{\sin \frac{n}{2} x}{\sin \frac{x}{2}}.$$

Thus

$$T = \sin \frac{n+1}{2} x \frac{\sin \frac{nx}{2}}{\sin \frac{x}{2}}.$$

61  $\frac{2(2 - \cos x)}{5 - 4 \cos x}.$

64 a)  $\frac{\sin \left( a + \frac{n-1}{2} h \right) \sin \frac{nh}{2}}{\cos \frac{h}{2}},$  if  $n$  is even,

$$\frac{\cos \left( a + \frac{n-1}{2} h \right) \cos \frac{nh}{2}}{\cos \frac{h}{2}},$$
 if  $n$  is odd;

b)  $\frac{\cos \left( a + \frac{n-1}{2} h \right) \sin \frac{nh}{2}}{\cos \frac{h}{2}},$  if  $n$  is even,

$$\frac{\sin \left( a + \frac{n-1}{2} h \right) \cos \frac{nh}{2}}{\cos \frac{h}{2}},$$
 if  $n$  is odd.

66 a)  $2^n \cos^n \frac{x}{2} \cos \frac{n+2}{2} x;$  b)  $2^n \cos^n \frac{x}{2} \sin \frac{n+2}{2} x.$

67 a)  $2^n \sin^n \frac{x}{2} \cos \frac{n\pi - (n+2)x}{2};$  b)  $2^n \sin^n \frac{x}{2} \sin \frac{(n+2)x - n\pi}{2}.$

68 The limit of the sum is equal to the vector corresponding to the number  $(3 + i)/5.$

$$69 \quad \frac{n}{2} - \frac{\sin 4nx}{4 \sin 2x}.$$

$$71 \quad a) \frac{3 \cos \frac{n+1}{2} x \sin \frac{nx}{2}}{4 \sin \frac{x}{2}} + \frac{\cos \frac{3(n+1)}{2} x \sin \frac{3nx}{2}}{4 \sin \frac{3x}{2}};$$

$$b) \frac{3 \sin \frac{n+1}{2} x \sin \frac{nx}{2}}{4 \sin \frac{x}{2}} - \frac{\sin \frac{3(n+1)}{2} x \sin \frac{3nx}{2}}{4 \sin \frac{3x}{2}}.$$

$$72 \quad a) \frac{(n+1) \cos nx - n \cos (n+1)x - 1}{4 \sin^2 \frac{x}{2}}; \quad b) \frac{(n+1) \sin nx - n \sin (n+1)x}{4 \sin^2 \frac{x}{2}}.$$

$$73 \quad e^a (\cos b + i \sin b).$$

$$75 \quad a) -3; \frac{3 \pm i \sqrt{3}}{2}; \quad b) -3; \frac{3 \pm 5i \sqrt{3}}{2};$$

$$c) -7; -1 \pm i \sqrt{3}; \quad d) -1; \frac{-5 \pm 5i \sqrt{3}}{2}; \quad e) 2; -1 \pm \sqrt{3};$$

$$f) \sqrt[3]{2} - \sqrt[3]{4}; \frac{\sqrt[3]{4} - \sqrt[3]{2}}{2} \pm \frac{i \sqrt{3}}{2} (\sqrt[3]{4} + \sqrt[3]{2});$$

$$g) \sqrt[3]{9} - 2\sqrt[3]{3}; \frac{2\sqrt[3]{3} - \sqrt[3]{9}}{2} \pm \frac{i \sqrt{3}}{2} (\sqrt[3]{9} + 2\sqrt[3]{3});$$

$$h) 1 - \sqrt[3]{2} - \sqrt[3]{4}; \frac{2 + \sqrt[3]{2} + \sqrt[3]{4}}{2} \pm \frac{i \sqrt{3}}{2} (\sqrt[3]{4} - \sqrt[3]{2});$$

$$i) -(1 + \sqrt[3]{3} + \sqrt[3]{9}); \frac{-2 + \sqrt[3]{3} + \sqrt[3]{9}}{2} \pm \frac{i \sqrt{3}}{2} (\sqrt[3]{9} - \sqrt[3]{3});$$

$$j) 2; -1 \pm 2i \sqrt{3}; \quad k) 2; -1 \pm 3i \sqrt{3}; \quad l) 2; -1 \pm 4i \sqrt{3};$$

$$m) 1; -2 \pm \sqrt{3}; \quad n) 4; -1 \pm 4i \sqrt{3}; \quad o) -2i; \quad i; \quad i;$$

$$p) -1 - i; \quad -1 - i; \quad 2 + 2i;$$

$$q) -(a + b); \frac{a + b}{2} \pm \frac{i \sqrt{3}}{2} (a - b);$$

$$r) -(a\sqrt[3]{f^2g} + b\sqrt[3]{fg^2});$$

$$\frac{a\sqrt[3]{f^2g} + b\sqrt[3]{fg^2}}{2} \pm \frac{i\sqrt{3}}{2}(a\sqrt[3]{f^2g} - b\sqrt[3]{fg^2});$$

$$s) 2.1149; -0.2541; -1.8608;$$

$$t) 1.5981; 0.5115; -2.1007.$$

76 Solution.

$$x_1 - x_2 = \alpha(1 - \omega) + \beta(1 - \omega^2) = (1 - \omega)(\alpha - \beta\omega^2)$$

$$x_1 - x_3 = \alpha(1 - \omega^2) + \beta(1 - \omega) = (1 - \omega^2)(\alpha - \beta\omega)$$

$$x_2 - x_3 = \alpha(\omega - \omega^2) + \beta(\omega^2 - \omega) = (\omega - \omega^2)(\alpha - \beta)$$

$$(x_1 - x_2)(x_1 - x_3)(x_2 - x_3) = 3(\omega - \omega^2)(\alpha^3 - \beta^3);$$

$$(x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2 =$$

$$= -27[(\alpha^3 + \beta^3)^2 - 4\alpha^3\beta^3] = -27q^2 - 4p^3.$$

77 Solution.

The cubic equation mentioned in the Hints is

$z^3 - 3(px + q)z + x^3 + p^3 - 3qx - 3pq = 0$ , which has the obvious root  $z = -(x + p)$ . The two other roots of the cubic equation are

$$z_{2,3} = \frac{x + p \pm \sqrt{-3(x - p)^2 + 12q}}{2}. \quad \text{From problem 76,}$$

part 1, the desired equation can be written in the form

$$-\frac{1}{27}(z_2 - z_3)^2(z_3 - z_1)^2(z_1 - z_2)^2 = -\frac{1}{27}[-3(x - p)^2 + 12q]$$

$$\times \left[ \frac{3(x + p) + \sqrt{-3(x - p)^2 + 12q}}{2} \right]^2 \left[ \frac{3(x + p) - \sqrt{-3(x - p)^2 + 12q}}{2} \right]^2 =$$

$$= [(x - p)^2 - 4q](x^2 + px + p^2 - q)^2,$$

the roots of which are easily seen to be:

$$x_{1,2} = p \pm 2\sqrt{q}; \quad x_3 = x_4 = \frac{-p + \sqrt{4q - 3p^2}}{2};$$

$$x_5 = x_6 = \frac{-p - \sqrt{4q - 3p^2}}{2}.$$

78 The left member can be written in the form

$$\alpha^5 + \beta^5 + 5(\alpha + \beta)(\alpha^2 + \alpha\beta + \beta^2 - a)(\alpha\beta - a) - 2b = 0.$$

Answer.  $x = \alpha + \beta$ , where

$$\alpha = \sqrt[5]{b + \sqrt{b^2 - a^5}}; \quad \beta = \sqrt[5]{b - \sqrt{b^2 - a^5}}; \quad \alpha\beta = a.$$

- 79 a)  $\pm\sqrt{2}$ ;  $1 \pm i\sqrt{3}$ ; b)  $-1 \pm \sqrt{6}$ ;  $\pm i\sqrt{3}$ ;  
 c)  $\pm\sqrt{2}$ ;  $\frac{1 \pm i\sqrt{3}}{2}$ ; d)  $\frac{1 \pm \sqrt{5}}{2}$ ;  $\frac{3 \pm \sqrt{5}}{2}$ ;  
 e)  $\frac{1 \pm \sqrt{13}}{2}$ ;  $1 \pm i$ ; f)  $\frac{1 \pm \sqrt{29}}{2}$ ;  $\frac{5 \pm i\sqrt{7}}{2}$ ;  
 g)  $\pm i$ ;  $1 \pm i\sqrt{2}$ ; h)  $\pm\sqrt{5}$ ;  $\frac{1 \pm i\sqrt{7}}{2}$ ;  
 i)  $\pm i$ ;  $-1 \pm i\sqrt{6}$ ; j)  $-2 \pm 2\sqrt{2}$ ;  $-1 \pm i$ ;  
 k) 1; 3;  $1 \pm \sqrt{2}$ ; l) 1; -1;  $1 \pm 2i$ ;  
 m)  $\frac{1 + \sqrt{5} \pm \sqrt{22 + 2\sqrt{5}}}{4}$ ;  $\frac{1 - \sqrt{5} \pm \sqrt{22 - 2\sqrt{5}}}{4}$ ;  
 n)  $\frac{1 + \sqrt{5} \pm \sqrt{30 - 6\sqrt{5}}}{4}$ ;  $\frac{1 - \sqrt{5} \pm \sqrt{30 + 6\sqrt{5}}}{4}$ ;  
 o)  $\frac{1 + \sqrt{2}}{2} \pm \frac{1}{2}\sqrt{-1 - 2\sqrt{2}}$ ;  $\frac{1 - \sqrt{2}}{2} \pm \frac{1}{2}\sqrt{-1 + 2\sqrt{2}}$ ;  
 p)  $1 + \sqrt{7} \pm \sqrt{6 + 2\sqrt{7}}$ ;  $1 - \sqrt{7} \pm \sqrt{6 - 2\sqrt{7}}$ ;  
 q)  $\frac{1 \pm \sqrt{4\sqrt{3} - 3}}{2}$ ;  $\frac{1 \pm \sqrt{-4\sqrt{3} - 3}}{2}$ ;  
 r)  $\frac{1 + \sqrt{5} \pm \sqrt{-2 - 6\sqrt{5}}}{4}$ ;  $\frac{1 - \sqrt{5} \pm \sqrt{-2 + 6\sqrt{5}}}{4}$ ;  
 s)  $\frac{1 + \sqrt{2} \pm \sqrt{-5 + 2\sqrt{2}}}{4}$ ;  $\frac{1 - \sqrt{2} \pm \sqrt{-5 - 2\sqrt{2}}}{4}$ ;  
 t)  $\frac{1 + \sqrt{3} \pm \sqrt{12 + 2\sqrt{3}}}{4}$ ;  $\frac{1 - \sqrt{3} \pm \sqrt{12 - 2\sqrt{3}}}{4}$ .

80 Solution.

$$\begin{aligned}
 & x^4 + ax^3 + bx^2 + cx + d = \\
 & = \left(x^2 + \frac{a}{2}x + \frac{\lambda}{2} + mx + n\right)\left(x^2 + \frac{a}{2}x + \frac{\lambda}{2} - mx - n\right); \\
 & x_1x_2 = \frac{\lambda}{2} + n; \quad x_3x_4 = \frac{\lambda}{2} - n; \quad \lambda = x_1x_2 + x_3x_4.
 \end{aligned}$$

81      a)  $\pm 1$ ; b)  $1$ ;  $-\frac{1}{2} \pm i \frac{\sqrt{3}}{2}$ ; c)  $\pm 1$ ;  $\pm i$ ;  
d)  $\pm 1$ ;  $\pm \frac{1}{2} \pm i \frac{\sqrt{3}}{2}$ ; e)  $\pm 1$ ;  $\pm i$ ;  $\pm \frac{\sqrt{2}}{2}(1 \pm i)$ ;  
f)  $\pm 1$ ;  $\pm i$ ;  $\pm \frac{1}{2} \pm i \frac{\sqrt{3}}{2}$ ;  $\pm \frac{\sqrt{3}}{2} \pm \frac{i}{2}$ ;  
g)  $\pm 1$ ;  $\pm i$ ;  $\pm \frac{1}{2} \pm i \frac{\sqrt{3}}{2}$ ;  $\pm \frac{\sqrt{2}}{2}(1 \pm i)$ ;  $\pm \frac{\sqrt{3}}{2} \pm \frac{i}{2}$ ;  
 $\pm \frac{\sqrt{6} + \sqrt{2}}{4} \pm i \frac{\sqrt{6} - \sqrt{2}}{4}$ ;  $\pm \frac{\sqrt{6} - \sqrt{2}}{4} \pm i \frac{\sqrt{6} + \sqrt{2}}{4}$ .

82      a)  $-1$ ; b)  $-\frac{1}{2} \pm i \frac{\sqrt{3}}{2}$ ; c)  $\pm i$ ; d)  $\frac{1}{2} \pm i \frac{\sqrt{3}}{2}$ ;  
e)  $\pm \frac{\sqrt{2}}{2}(1 \pm i)$ ; f)  $\pm \frac{\sqrt{3}}{2} \pm \frac{i}{2}$ ;  
g)  $\pm \frac{\sqrt{6} + \sqrt{2}}{4} \pm i \frac{\sqrt{6} - \sqrt{2}}{4}$ ;  $\pm \frac{\sqrt{6} - \sqrt{2}}{4} \pm i \frac{\sqrt{6} + \sqrt{2}}{4}$ .

83      a) 20; 20; 180;      b) 72; 144; 12.

84       $\cos \frac{2k\pi}{7} + i \sin \frac{2k\pi}{7}$ ,      for  $k = 1, 2, 3, 4, 5, 6$ .

85      a) Set  $\epsilon_k = \cos \frac{2k\pi}{16} + i \sin \frac{2k\pi}{16}$ . Then we find:

$\epsilon_0$  has index 1;

$\epsilon_8$  has index 2;

$\epsilon_4, \epsilon_{12}$  have index 4;

$\epsilon_2, \epsilon_6, \epsilon_{10}, \epsilon_{14}$  have index 8;



the remaining 16-th roots of unity, namely

$\epsilon_1, \epsilon_3, \epsilon_5, \epsilon_7, \epsilon_9, \epsilon_{11}, \epsilon_{13}, \epsilon_{15}$  are primitive.

b) Set  $\epsilon_k = \cos \frac{2k\pi}{20} + i \sin \frac{2k\pi}{20}$ . Then we find:

$\epsilon_0$  has index 1;

$\epsilon_{10}$  has index 2;

$\epsilon_5, \epsilon_{15}$  have index 4;

$\epsilon_4, \epsilon_8, \epsilon_{12}, \epsilon_{16}$  have index 5;

$\epsilon_2, \epsilon_6, \epsilon_{14}, \epsilon_{18}$  have index 10;

the remaining 20-th roots of unity, namely

$\epsilon_1, \epsilon_3, \epsilon_7, \epsilon_9, \epsilon_{11}, \epsilon_{13}, \epsilon_{17}, \epsilon_{19}$  are primitive.

c) Set  $\epsilon_k = \cos \frac{2k\pi}{24} + i \sin \frac{2k\pi}{24}$ . Then we find:

$\epsilon_0$  has index 1;

$\epsilon_{12}$  has index 2;

$\epsilon_8, \epsilon_{16}$  have index 3;

$\epsilon_6, \epsilon_{18}$  have index 4;

$\epsilon_4, \epsilon_{20}$  have index 6;

$\epsilon_3, \epsilon_9, \epsilon_{15}, \epsilon_{21}$  have index 8;

$\epsilon_2, \epsilon_{10}, \epsilon_{14}, \epsilon_{22}$  have index 12;

the remaining 24-th roots of unity, namely

$\epsilon_1, \epsilon_5, \epsilon_7, \epsilon_{11}, \epsilon_{13}, \epsilon_{17}, \epsilon_{19}, \epsilon_{23}$  are primitive.

86

a)  $X_1(x) = x - 1$ ; b)  $X_2(x) = x + 1$ ;

c)  $X_3(x) = x^2 + x + 1$ ; d)  $X_4(x) = x^2 + 1$ ;

e)  $X_5(x) = x^4 + x^3 + x^2 + x + 1$ ; f)  $X_6(x) = x^2 - x + 1$ ;

g)  $X_7(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ ; h)  $X_8(x) = x^4 + 1$ ;

i)  $X_9(x) = x^6 + x^3 + 1$ ; j)  $X_{10}(x) = x^4 - x^3 + x^2 - x + 1$ ;

k)  $X_{11}(x) = x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 +$   
 $+ x^2 + x + 1$ ; l)  $X_{12}(x) = x^4 - x^2 + 1$ ;

$$m) X_{15}(x) = x^8 - x^7 + x^5 - x^4 + x^3 - x + 1;$$

$$n) X_{105}(x) = x^{48} + x^{47} + x^{46} - x^{43} - x^{42} - 2x^{41} - x^{40} \\ - x^{39} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} - x^{28} - x^{26} \\ - x^{24} - x^{22} - x^{20} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} \\ - x^9 - x^8 - 2x^7 - x^6 - x^5 + x^2 + x + 1.$$

$$87 \quad 2/(1 - \epsilon).$$

$$88 \quad 0, \text{ if } n > 1.$$

$$89 \quad n, \text{ if } k \text{ is divisible by } n; 0, \text{ if } k \text{ is not divisible by } n.$$

$$90 \quad m(x^m + 1).$$

$$91 \quad -n/(1 - \epsilon), \text{ if } \epsilon \neq 1; n(n + 1)/2, \text{ if } \epsilon = 1.$$

$$92 \quad -\frac{n^2(1 - \epsilon) + 2n}{(1 - \epsilon)^2}, \text{ if } \epsilon \neq 1; \frac{n(n + 1)(2n + 1)}{6}, \text{ if } \epsilon = 1.$$

$$93 \quad a) -n/2; \quad b) -\frac{1}{2} n \cotan(\pi/n).$$

$$94 \quad a) 1; \quad b) 0; \quad c) -1.$$

$$95 \quad x_0 = 1;$$

$$x_1 = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5} = \frac{\sqrt{5}-1}{4} + \frac{i}{4} \sqrt{10+2\sqrt{5}};$$

$$x_2 = \cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5} = -\frac{\sqrt{5}+1}{4} + \frac{i}{4} \sqrt{10-2\sqrt{5}};$$

$$x_3 = \cos \frac{6\pi}{5} + i \sin \frac{6\pi}{5} = -\frac{\sqrt{5}+1}{4} - \frac{i}{4} \sqrt{10-2\sqrt{5}};$$

$$x_4 = \cos \frac{8\pi}{5} + i \sin \frac{8\pi}{5} = \frac{\sqrt{5}-1}{4} - \frac{i}{4} \sqrt{10+2\sqrt{5}}.$$

$$96 \quad \sin 18^\circ = \frac{\sqrt{5}-1}{4}; \quad \cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}.$$

- 97 Solution. Divide the equation  $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$  by  $x^3$ . Using easy transformations, this leads to the equation

$$\left(x + \frac{1}{x}\right)^3 + \left(x + \frac{1}{x}\right)^2 - 2\left(x + \frac{1}{x}\right) - 1 = 0.$$

Now the equation  $z^3 + z^2 - 2z - 1 = 0$  is satisfied by  $z = 2 \cos \frac{4\pi}{7} = -2 \sin \frac{\pi}{14}$ . Moreover  $t = 2 \sin \frac{\pi}{14}$  satisfies the equation  $t^3 - t^2 - 2t + 1 = 0$ .

These equations are the simplest possible ones in the sense that every other equation that has rational coefficients and has a root in common with these must be of higher degree. The proof of this assertion is most conveniently carried through by advanced methods.

- 98 Solution. Set  $n = 2m$ ; the equation  $x^n - 1 = 0$  has the two real roots 1, -1; and  $2m - 2$  complex roots. Indeed  $\varepsilon_k = \cos \frac{2k\pi}{2m} + i \sin \frac{2k\pi}{2m}$  is conjugate to  $\varepsilon_{2m-k} = \cos \frac{2(2m-k)\pi}{2m} + i \sin \frac{2(2m-k)\pi}{2m}$ . Thus we obtain

$$\begin{aligned} x^{2m} - 1 &= (x^2 - 1)(x - \varepsilon_1)(x - \bar{\varepsilon}_1)(x - \varepsilon_2)(x - \bar{\varepsilon}_2) \dots \\ &\quad \dots (x - \varepsilon_{m-1})(x - \bar{\varepsilon}_{m-1}); \quad x^{2m} - 1 \\ &= (x^2 - 1)[x^2 - (\varepsilon_1 + \bar{\varepsilon}_1)x + 1] \dots [x^2 - (\varepsilon_{m-1} + \bar{\varepsilon}_{m-1})x + 1]; \\ x^{2m} - 1 &= (x^2 - 1) \prod_{k=1}^{m-1} \left(x^2 - 2x \cos \frac{k\pi}{m} + 1\right). \end{aligned}$$

If  $n = 2m + 1$ , then we obtain

$$x^{2m+1} - 1 = (x - 1) \prod_{k=1}^m \left(x^2 - 2x \cos \frac{2k\pi}{2m+1} + 1\right).$$

by using similar reasoning.

99 Solution.

a) We have

$$\frac{x^{2m}-1}{x^2-1} = \prod_{k=1}^{m-1} \left( x^2 - 2x \cos \frac{k\pi}{m} + 1 \right).$$

Setting  $x = 1$ , we find  $m = 2^{m-1} \prod_{k=1}^{m-1} \left( 1 - \cos \frac{k\pi}{m} \right),$

or,  $m = 2^{2(m-1)} \prod_{k=1}^{m-1} \sin^2 \frac{k\pi}{2m},$  . Finally

$$\frac{\sqrt{m}}{2^{m-1}} = \prod_{k=1}^{m-1} \sin \frac{k\pi}{2m}.$$

The proof of formula b) is similar.

100 Solution. Set  $x = -a/b$  in the identity

$$x^n - 1 = \prod_{k=0}^{n-1} (x - \varepsilon_k) \quad , \quad \varepsilon_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$$

We find  $(-1)^n \frac{a^n}{b^n} - 1 = (-1)^n \prod_{k=0}^{n-1} \left( \frac{a}{b} + \varepsilon_k \right) \quad , \text{ etc.}$

101 We start with the obvious factorization

$$\begin{aligned} \cos n\theta + i \sin n\theta - 1 &= \prod_{k=0}^{n-1} (\cos \theta + i \sin \theta - \varepsilon_k), \\ \cos n\theta - i \sin n\theta - 1 &= \prod_{k=0}^{n-1} (\cos \theta - i \sin \theta - \varepsilon_k). \end{aligned}$$

Finally we obtain the required result by multiplying these two relations.

102 Solution.

$$\begin{aligned}
\prod_{k=0}^{n-1} \frac{(t + \varepsilon_k)^n - 1}{t} &= \frac{1}{t^n} \prod_{k=0}^{n-1} \prod_{s=0}^{n-1} (t + \varepsilon_k - \varepsilon_s) = \\
&= \frac{1}{t^n} \prod_{k=0}^{n-1} \prod_{s=0}^{n-1} \left[ t - \varepsilon_k \left( \frac{\varepsilon_s}{\varepsilon_k} - 1 \right) \right] = \frac{1}{t^n} \prod_{k=0}^{n-1} \prod_{s=0}^{n-1} [t - \varepsilon_k (\varepsilon_s - 1)] = \\
&= \frac{1}{t^n} \prod_{s=0}^{n-1} \prod_{k=0}^{n-1} [t - \varepsilon_k (\varepsilon_s - 1)] = \frac{1}{t^n} \prod_{s=0}^{n-1} [t^n - (\varepsilon_s - 1)^n] = \\
&= \prod_{k=1}^{n-1} [t^n - (\varepsilon_k - 1)^n].
\end{aligned}$$

103 From the relation  $|x| = |x|^{n-1}$ , it follows that  $|x| = 0$  or  $|x| = 1$ . If  $|x| = 0$ , then  $x = 0$ . If  $|x| = 1$ , then  $x\bar{x} = 1$ .

On the other hand,  $x\bar{x} = x^n$ . Therefore  $x^n = 1$ .

Thus  $x = 0$  or  $x = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$ ,

$k = 0, 1, 2, \dots, n-1$ . The converse proposition is easy.

104 Solution. If  $z$  satisfies the given equation then

$$\left| \frac{z-a}{z-b} \right| = \sqrt[n]{\left| \frac{\mu}{\lambda} \right|}. \quad \text{The geometric locus of points,}$$

the distances of which to two given points lie in a given ratio, is a circle (in special cases a straight line).

105 a) We have  $\frac{x+1}{x-1} = \varepsilon_k$ , where

$\varepsilon_k = \cos \frac{2k\pi}{m} + i \sin \frac{2k\pi}{m}$ ,  $k = 1, 2, \dots, m-1$ . Moreover

$x = \frac{\varepsilon_k + 1}{\varepsilon_k - 1}$ . This last relation can be transformed

into the form  $x_k = i \operatorname{ctg} \frac{k\pi}{m}$ ,  $k = 1, 2, \dots, m-1$ ;

$$b) \quad x_k = \operatorname{ctg} \frac{k\pi}{m}, \quad k = 1, 2, \dots, m-1;$$

$$c) \quad x_k = \frac{a}{\varepsilon_k \sqrt[n]{2-1}},$$

$$\text{where } \varepsilon_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}, \quad k = 0, 1, 2, \dots, n-1.$$

106 Solution. Set  $A = \cos \varphi + i \sin \varphi$ . Then  $\frac{1+ix}{1-ix} = \eta_k^2$ ,  
 where  $\eta_k = \cos \frac{\varphi+2k\pi}{2m} + i \sin \frac{\varphi+2k\pi}{2m}$ ,  $k = 0, 1, \dots, m-1$ .

$$\text{Thus } x = \frac{\eta_k^2 - 1}{i(\eta_k^2 + 1)} = \frac{\eta_k - \eta_k^{-1}}{i(\eta_k + \eta_k^{-1})} = \operatorname{tg} \frac{\varphi + 2k\pi}{2m}.$$

107 Solution. Proceeding as suggested in the Hints  
 we obtain

$$S + Ti = \mu(1 + \lambda x)^n, \quad S - Ti = \bar{\mu}(1 + \bar{\lambda}x)^n,$$

where  $\lambda = \cos \alpha + i \sin \alpha$ ,  $\mu = \cos \varphi + i \sin \varphi$ . Thus

$$2S = \mu(1 + \lambda x)^n + \bar{\mu}(1 + \bar{\lambda}x)^n.$$

The equation now takes the form

$$\mu(1 + \lambda x)^n + \bar{\mu}(1 + \bar{\lambda}x)^n = 0; \quad ;$$

$$x_k = - \frac{\sin \frac{(2k+1)\pi - 2\varphi}{2n}}{\sin \frac{(2k+1)\pi - 2\varphi - 2n\alpha}{2n}}; \quad k = 0, 1, 2, \dots, n-1.$$

108 Solution. Set  $\alpha^a = 1$ ;  $\beta^b = 1$ . Then

$$(\alpha\beta)^{ab} = (\alpha^a)^b \cdot (\beta^b)^a = 1.$$

109 Solution. Let  $\epsilon$  be a common zero of the  
 polynomials  $x^a - 1$ ,  $x^b - 1$ ; let  $s$  be the exponent  
 to which  $\epsilon$  belongs. Then  $s$  is a common divisor  
 of  $a$ ,  $b$ . Thus  $s$  must be 1, and  $\epsilon = 1$ . The  
 converse is obvious.

110 Solution. Let  $\alpha_k$ ,  $\beta_s$  be the  $a$ -th and  $b$ -th roots of 1;  $k = 0, 1, 2, \dots, a-1$ ;  $s = 0, 1, 2, \dots, b-1$ . The result of problem 108 shows that it will be sufficient to establish the fact that all the products  $\alpha_k \beta_s$  are distinct. Suppose on the contrary that  $\alpha_{k_1} \beta_{s_1} = \alpha_{k_2} \beta_{s_2}$ . Then  $\frac{\alpha_{k_1}}{\alpha_{k_2}} = \frac{\beta_{s_2}}{\beta_{s_1}}$ , i.e.  $\alpha_i = \beta_j$ . Now use problem 109 and conclude that  $\alpha_i = \beta_j = 1$ , i.e.  $k_1 = k_2$ ;  $s_1 = s_2$ .

111 Solution. Let  $\alpha, \beta$  be primitive  $a$ -th and  $b$ -th roots of 1. Set  $(\alpha\beta)^s = 1$ . Then  $\alpha^{bs} = 1$ ;  $\beta^{as} = 1$ . This shows that  $bs$  is divisible by  $a$ , and  $as$  is divisible by  $b$ . Therefore  $s$  is divisible by  $ab$ .

Let  $\lambda$  be a primitive  $ab$ -th root of 1. Then  $\lambda = \alpha^k \beta^s$  by problem 110. Suppose the exponent to which  $\alpha^k$  belongs is  $a_1 < a$ . Then  $\lambda^{a_1 b} = (\alpha^k)^{a_1 b} (\beta^s)^{a_1 b} = 1$ , a contradiction. In the same way it can be shown that  $\beta^s$  is a primitive  $b$ -th root of 1.

112 Follows immediately from problem 111.

113 As suggested in the Hints, count all numbers not exceeding  $p^a$  that are multiples of  $p$ . These are  $1 \cdot p, 2 \cdot p, 3 \cdot p, \dots, p^{a-1} \cdot p$ . The number of these is clearly  $p^{a-1}$ . Thus  $\varphi(p^a) = p^a - p^{a-1} = p^a \left(1 - \frac{1}{p}\right)$ . From problem 112,

$$\varphi(n) = \varphi(p_1^{a_1}) \varphi(p_2^{a_2}) \dots \varphi(p_k^{a_k}) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \dots \left(1 - \frac{1}{p_k}\right).$$

- 114 Solution. If  $\epsilon$  is a primitive  $n$ -th root of 1, then the conjugate complex number  $\bar{\epsilon}$  is also a primitive  $n$ -th root of 1. Furthermore, since  $n > 2$ ,  $\epsilon \neq \pm 1$ .

115  $X_p(x) = x^{p-1} + x^{p-2} + \dots + x + 1.$

116  $X_{p^m}(x) = x^{(p-1)p^{m-1}} + x^{(p-2)p^{m-1}} + \dots + x^{p^{m-1}} + 1.$

- 117 The way to carry out the suggestion in the Hints is to base the arguments on problem 111.

Let  $\alpha_1, \alpha_2, \dots, \alpha_{\varphi(n)}$  be the primitive  $n$ -th roots of 1. Then  $-\alpha_1, -\alpha_2, \dots, -\alpha_{\varphi(n)}$  are primitive  $2n$ -th roots of 1. Thus

$$\begin{aligned} X_{2n}(x) &= (x + \alpha_1)(x + \alpha_2) \dots (x + \alpha_{\varphi(n)}) = \\ &= (-1)^{\varphi(n)} (-x - \alpha_1) \dots (-x - \alpha_{\varphi(n)}), \end{aligned}$$

or, by problem 114,  $X_{2n}(x) = X_n(-x).$

- 118 Solution. Let the primitive  $nd$ -th roots of 1 be  $\epsilon_k = \cos \frac{2k\pi}{nd} + i \sin \frac{2k\pi}{nd}$ , that is, let  $k, n$  be relatively prime. By the Euclidean algorithm, we can write  $k = nq + r$ ,  $0 < r < n$ . Thus

$$\epsilon_k = \cos \frac{2q\pi + \frac{2r\pi}{n}}{d} + i \sin \frac{2q\pi + \frac{2r\pi}{n}}{d}.$$

This means that  $\epsilon_k$  is one of the  $n$ -th roots of  $\eta_r = \cos \frac{2r\pi}{n} + i \sin \frac{2r\pi}{n}$ , where  $\eta_r$  is a primitive  $n$ -th root of 1; indeed every common factor of  $r, n$  is a common factor of  $k, n$ .

To establish the converse, let  $\eta_r = \cos \frac{2r\pi}{n} + i \sin \frac{2r\pi}{n}$  be a primitive  $n$ -th root of 1, so that  $r, n$  are relatively prime.



$$\text{The numbers } \varepsilon_q = \cos \frac{2q\pi + \frac{2r\pi}{n}}{d} + i \sin \frac{2q\pi + \frac{2r\pi}{n}}{d} =$$

$$\cos \frac{2\pi(r+ng)}{nd} + i \sin \frac{2\pi(r+ng)}{nd}, \quad q = 0, 1, 2, \dots, d-1$$

are primitive  $nd$ -th roots of 1. For if the numbers  $r + mq$ ,  $nd$  had a common prime divisor  $p$ , then  $p|n$ ,  $p|r$ , a contradiction.

119 Solution. Let  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{\varphi(n')}$  be primitive  $n'$ -th roots of 1. Then  $X_{n'}(x^{n''}) = \prod_{k=1}^{\varphi(n')} (x^{n''} - \varepsilon_k)$ .

If we now factor  $x^{n''} - \varepsilon_k$  :

$$x^{n''} - \varepsilon_k = (x - \varepsilon_{k,1})(x - \varepsilon_{k,2}) \cdots (x - \varepsilon_{k,n''})$$

we can write

$$X'_{n'}(x^{n''}) = \prod_{\substack{k=1 \\ i=1}}^{k=\varphi(n')} (x - \varepsilon_{k,i}).$$

The result of problem 118 shows that every factor

$x - \varepsilon_{k,i}$  is a divisor of  $X_n(x)$  and conversely.

Thus the degrees of  $X_n(x)$ ,  $X_{n'}(x^{n''})$  are equal, for

$$\varphi(n) = n''\varphi(n'),$$

121 Solution. The sum of all the  $n$ -th roots of 1 is zero. But every  $n$ -th root of 1 belongs to an exponent  $d$  that divides  $n$ , and conversely. Thus

$$\sum_{d|n} \mu(d) = 0.$$

122 Solution. If  $\varepsilon_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$  belongs to the exponent  $n_1$ , the factor  $x - \varepsilon_k$  is a divisor of the binomials  $x^d - 1$  for which  $d|n_1$ , and not for those for which  $d \nmid n_1$ . Moreover if  $d$  runs through all

divisors of  $n$  divisible by  $n_1$ , then  $n/d$  runs through all divisors of  $n/n_1$ . Therefore  $x - \epsilon_k$  appears with exponent  $\sum_{d/n_1} \mu(d_1)$  in the right member. This sum has the value 0 if  $n/n_1 \neq 1$ ; it has the value 1 if  $n = n_1$ .

- 123 Solution. If  $n = p^a$  for  $p$  prime, then  $X_n(1) = p$ . If  $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ , for  $p_1, p_2, \dots, p_k$  distinct primes, then by problem 119,  $X_n(1) = X_{n'}(1)$  where  $n' = p_1 p_2 \dots p_k$ .

Now set  $n = p_1 p_2 \dots p_k$ ;  $k \geq 2$ ;  $n_1 = n/p_k$ . Note that one can write down all the divisors of  $n$  by multiplying the distinct divisors of  $n_1$  and their products by  $p_k$ . Therefore

$$\begin{aligned} X_n(x) &= \prod_{d/n} (x^d - 1)^{\mu\left(\frac{n}{d}\right)} = \prod_{d/n_1} (x^d - 1)^{\mu\left(\frac{n}{d}\right)} \cdot \prod_{d/n_1} (x^{dp_k} - 1)^{\mu\left(\frac{n}{dp_k}\right)} \\ &= [X_{n_1}(x)]^{-1} \cdot X_{n_1}(x^{p_k}). \end{aligned}$$

Thus  $X_n(x) = 1$ .

- 124 Solution. 1) If  $n$  is odd and greater than 1, problem 117 shows that  $X_n(-1) = X_{2n}(1) = 1$ .

$$2) \text{ If } n = 2^k, \text{ then } X_n = \frac{x^n - 1}{x^{\frac{n}{2}} - 1} = x^{\frac{n}{2}} + 1.$$

But  $X_n(-1) = 0$  if  $k = 1$ ;  $X_n(-1) = 2$  if  $k > 1$ .

3) Let  $n_1 > 1$  be odd;  $n = 2n_1$ . By problem 117,  $X_n(-1) = X_{n_1}(1)$  and therefore  $X_n(-1)$  will have the value  $p$  if  $n_1 = p^{\alpha}$  ( $p$  prime), or will have the value 1 if  $n_1 \neq p^{\alpha}$ .

4) Let  $n = 2^k n_1$ ,  $k > 1$ ,  $n_1 = p_1^{a_1} p_2^{a_2} \dots p_s^{a_s}$ , where  $p_1, p_2, \dots, p_s$  are distinct odd primes. By problem 119,  $X_n(x) = X_{2^{p_1} p_2 \dots p_s}(x^\lambda)$ , where  $\lambda = 2^{k-1} p_1^{a_1-1} \dots p_s^{a_s-1}$ . Thus it follows that  $X_n(-1) = X_n(1) = 1$ .

125 Solution. Let  $\epsilon_1, \epsilon_2, \dots, \epsilon_{\varphi(n)}$  be the primitive  $n$ -th roots of 1. Then:

$$s = \epsilon_1 \epsilon_2 + \epsilon_1 \epsilon_3 + \dots + \epsilon_{\varphi(n)-1} \cdot \epsilon_{\varphi(n)} = \frac{[\mu(n)]^2 - (\epsilon_1^2 + \epsilon_2^2 + \dots + \epsilon_{\varphi(n)}^2)}{2}.$$

1) If  $n$  is odd, then  $\epsilon_i^2$  is a primitive  $n$ -th root of 1, and  $\epsilon_i^2 = \epsilon_j^2$  only for  $i = j$ . Therefore

$$\epsilon_1^2 + \epsilon_2^2 + \dots + \epsilon_{\varphi(n)}^2 = \mu(n), \quad s = \frac{[\mu(n)]^2 - \mu(n)}{2}.$$

2) Let  $n_1$  be odd,  $n = 2n_1$ . By problem 111,  $\epsilon_1$  is a primitive  $n_1$ -th root of 1 and therefore by the paragraph above,  $\epsilon_1^2 + \epsilon_2^2 + \dots + \epsilon_{\varphi(n)}^2 = \mu(n_1) = -\mu(n)$ . Thus,  $s = \frac{[\mu(n)]^2 + \mu(n)}{2}$ .

3) Let  $n_1$  be odd,  $k > 1$ ,  $n = 2^k n_1$ . Then  $\epsilon_i^2$  belongs to the exponent  $n/2$ . By problem 118 it follows that  $\epsilon_1, \epsilon_2, \dots, \epsilon_{\varphi(n)}$  are the square roots of  $\eta_1, \eta_2, \dots, \eta_{\varphi(\frac{n}{2})}$ , and  $\eta_1, \eta_2, \dots, \eta_{\varphi(\frac{n}{2})}$  are primitive  $n/2$ -th roots of 1. Therefore,

$$\epsilon_1^2 + \epsilon_2^2 + \dots + \epsilon_{\varphi(n)}^2 = 2\left(\eta_1 + \eta_2 + \dots + \eta_{\varphi(\frac{n}{2})}\right) = 2\mu\left(\frac{n}{2}\right); \quad s = -\mu\left(\frac{n}{2}\right).$$

126 Solution.

$$S = \sum_{x=0}^{n-1} \varepsilon^{x^2} = \sum_{x=y}^{y+n-1} \varepsilon^{x^2} = \sum_{s=0}^{n-1} \varepsilon^{(y+s)^2} \quad \text{for}$$

any integer  $y$ ;

$$\begin{aligned} S' &= \sum_{y=0}^{n-1} \varepsilon^{-y^2}; \quad S'S = \sum_{y=0}^{n-1} \varepsilon^{-y^2} S = \sum_{y=0}^{n-1} \left( \varepsilon^{-y^2} \cdot \sum_{s=0}^{n-1} \varepsilon^{(y+s)^2} \right) = \\ &= \sum_{y=0}^{n-1} \sum_{s=0}^{n-1} \varepsilon^{2ys+s^2} = \sum_{s=0}^{n-1} \left( \varepsilon^{s^2} \cdot \sum_{y=0}^{n-1} \varepsilon^{2ys} \right) = n + \sum_{s=1}^{n-1} \varepsilon^{s^2} \cdot \sum_{y=0}^{n-1} (\varepsilon^{2s})^y = n \end{aligned}$$

for  $n$  odd;

$$SS' = n + n\varepsilon^{\left(\frac{n}{2}\right)^2} = n \left[ 1 + (-1)^{\frac{n}{2}} \right]$$

for  $n$  even. (Since  $\sum_{y=0}^{n-1} \varepsilon^{2sy} = 0$  for  $n \nmid 2s$ .)

Thus,  $|S| = \sqrt{n}$ , if  $n$  is odd;

$$|S| = \sqrt{n \left[ 1 + (-1)^{\frac{n}{2}} \right]},$$

if  $n$  is even.

CHAPTER II - SOLUTIONS  
COMPUTATION OF DETERMINANTS

127      a) 5; b) 5; c) 1; d)  $ab - c^2 - d^2$ ; e)  $\alpha^2 + \beta^2 - \gamma^2 - \delta^2$ ;  
 f)  $\sin(\alpha - \beta)$ ; g)  $\cos(\alpha + \beta)$ ; h)  $\sec^2 \alpha$ ; i)  $-2$ ; j) 0; k)  $(b - c)(d - a)$ ;  
 l)  $4ab$ ; m)  $-1$ ; n)  $-1$ ; o)  $\frac{1 + i\sqrt{3}}{2}$ .

128      a) 1; b) 2; c)  $2a^2(a + x)$ ; d) 1; e)  $-2$ ; f)  $-2 - \sqrt{2}$ ;  
 g)  $-3i\sqrt{3}$ ; h)  $-3$ .

129      The number of transpositions is odd.

130      a) 10; b) 18; c) 36.

131      a)  $i = 8$ ;  $k = 3$ ;      b)  $i = 3$ ;  $k = 6$ .

132       $C_2^n$

133       $C_2^n - I$ .

134      a)  $\frac{n(n-1)}{2}$ ; b)  $\frac{n(n+1)}{2}$ .

135      a)  $\frac{n(3n+1)}{2}$ ; b)  $\frac{3n(n-1)}{2}$ .

136      This is easily proved by considering a pair of elements. Without loss of generality consider the first two  $a_1, a_2$ . If these are in their natural order then the permutation in the inverse direction keeps the number 1, 2 in their natural order. Similarly for the pair of numbers  $a_1, a_k$ .

On the other hand, if the numbers  $a_1, a_2$  are not in their natural order then the permutation going in the opposite direction involves an inversion to reach the order 1, 2. Similarly for the numbers  $a_1, a_k$ .

137 The word algorithm can be obtained from the word logarithm by an even number (4) of transpositions. Hence the permutations are of the same character. Both permutations are odd.

138 a) + ; b) + .

139 a) no; b) yes.

140  $i = 1$ ;  $k = 4$ .

141  $a_{11}a_{23}a_{32}a_{44}, a_{12}a_{23}a_{34}a_{41}, a_{14}a_{23}a_{31}a_{42}$ .

142  $-a_{14}a_{32} \det \begin{bmatrix} a_{31} & a_{32} & a_{35} \\ a_{41} & a_{42} & a_{45} \\ a_{51} & a_{52} & a_{55} \end{bmatrix}$ .

143 + .

144  $(-1)^e, e = C_2^n$ .

146 2; -1.

147 a)  $n!$ ; b)  $(-1)^{\frac{n(n-1)}{2}}$ ; c)  $n!$

148 a)  $(-1)^{\frac{n(n+1)}{2}}(n!)^{n+1}$ ; b)  $(-1)^{\frac{n(n+1)}{2}}(n!)^{n+1}$ .

149 Solution. Write each row as a column. On the one hand, the value of the determinant is unchanged; on the other hand, it is replaced by its complex conjugate.

150 Solution. Write each row as a column. On the one hand, the value of the determinant is unchanged; on the other hand, it is multiplied by  $-1$ .

151  $(-1)^{n-1} \Delta$ .

152 Multiply by  $(-1)^{\frac{n(n-1)}{2}}$ .

153 0, since the number of even permutations on  $n$  symbols is the same as the number of odd permutations.

154 a)  $x_1 = a_1; x_2 = a_2; \dots; x_{n-1} = a_{n-1};$

b)  $x_1 = 0; x_2 = 1; \dots; x_{n-1} = n - 2.$

c)  $x_1 = a_1; x_2 = a_2; \dots; x_{n-1} = a_{n-1}.$

156 0.

158  $(mq - np) \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$

159 a)  $a_1 a_2 \dots a_n \left( \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right);$

b)  $(-1)^{\frac{n(n-1)}{2}} a_1 a_2 \dots a_n \left( \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right).$

160  $3a - b + 2c + d.$  165 48.

161  $4t - x - y - z.$  166 1.

162  $2a - b - c - d.$  167 160.

163 -1,487,600. 168 12.

164 -29,400,000. 169 900.

170 394.

171 665.

- 172  $a^2 + b^2 + c^2 - 2(bc + ca + ab)$ . 173  $-2(x^3 + y^3)$ .  
 174  $(x+1)(x^2 - x + 1)^2$ . 175  $x^2 z^2$ . 176  $-3(x^2 - 1)(x^2 - 4)$ .  
 177  $\sin(c-a)\sin(c-b)\sin(a-b)$ . 178  $(af - be + cd)^2$ . 179  $n!$   
 180  $b_1 b_2 \dots b_n$ . 181  $(x - x_1)(x - x_2) \dots (x - x_n)$ .  
 182  $(n-1)!$  183  $-2(n-2)!$  184  $1$ .  
 185  $\frac{na^{n-1}}{2} [2a + (n-1)] h$ . 186  $\frac{na^{n-1}}{2} [2a + (n-1)h]$ .  
 187  $(-1)^{\frac{n(n+1)}{2}} [a_0 - a_1 + a_2 - \dots + (-1)^n a_n]$ .  
 188  $a_0 x^n + a_1 x^{n-1} + \dots + a_n$ . 189  $\frac{nx^n}{x-1} - \frac{x^n - 1}{(x-1)^2}$ .  
 190  $(n+1)! x^n$ . 191  $(x - a_1)(x - a_2) \dots (x - a_n)$ .  
 192  $[x + (n-1)a](x - a)^{n-1}$ . 193  $\frac{(x+a)^n + (x-a)^n}{2}$ .  
 194  $(-1)^n (n+1) a_1 a_2 \dots a_n$ .  
 195  $a_1 a_2 \dots a_n \left(1 + \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}\right)$ .  
 196  $h(x+h)^n$ . 197  $(-1)^{n-1} (n-1) x^{n-2}$ .  
 198  $(-1)^n 2^{n-1} a_1 a_2 \dots a_n \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}\right)$ .  
 199  $(-1)^{\frac{n(n-1)}{2}} \frac{n^{n-1} (n+1)}{2}$ . 200  $\frac{(n+1)^{n+1}}{n^n}$ .  
 201  $\prod_{k=1}^n (1 - ax_{k,k})$ . 202  $(-1)^{n-1} 2^{n-2} (n+1)$ .  
 203  $(-1)^n (a_0 b_0 + a_1 b_1 + \dots + a_n b_n) b_1 b_2 \dots b_{n-1}$ .  
 204  $a(a+b)(a+2b) \dots [a + (n+1)b]$ .

205  $x^n + (-1)^{n-1} y^n$ .

206  $0$ , if  $n > 2$ .

207  $0$ , if  $n > 2$ .



208 For  $n = 2$ , we use the hints as follows:

$$\begin{vmatrix} 1+a_1+x_1 & a_1+x_2 \\ a_2+x_1 & 1+a_2+x_2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & a_1+x_2 \\ 0 & a_2+x_2 \end{vmatrix} + \begin{vmatrix} a_1+x_1 & 0 \\ a_2+x_1 & 1 \end{vmatrix} \\ + \begin{vmatrix} a_1+x_1 & a_1+x_2 \\ a_2+x_1 & a_2+x_2 \end{vmatrix} = 1 + [(a_1+x_1) + (a_2+x_2)] + (a_2-a_1)(x_1-x_2).$$

In the general case, the  $n$ -th order determinant can be written as the sum of  $2^n$  determinants; one of these has the value 1;  $n$  have the values  $a_1 + x_1, a_2 + x_2, \dots, a_n + x_n$ ;  $n(n-1)/2$  have the respective values

$$(a_i - a_k)(x_i - x_k), \quad i > k.$$

The remaining terms have the value 0. Thus the result is

$$1 + \sum_{i=1}^n (a_i + x_i) + \sum_{i>k} (a_i - a_k)(x_i - x_k),$$

which can be written in the form

$$(1 + a_1 + a_2 + \dots + a_n)(1 + x_1 + x_2 + \dots + x_n) - n(a_1x_1 + a_2x_2 + \dots + a_nx_n).$$

209 0, if  $p > 2$ .

$$211 \quad \frac{n+1}{1-x} + \frac{x^{n+1}-1}{(1-x)^2}.$$

212 Solution. It is clear that  $\Delta_2 = x_1x_2\left(1 + \frac{a_1}{x_1} + \frac{a_2}{x_2}\right)$ . Suppose

$$\Delta_{n-1} = x_1x_2 \dots x_{n-1} \left(1 + \frac{a_1}{x_1} + \frac{a_2}{x_2} + \dots + \frac{a_{n-1}}{x_{n-1}}\right).$$

Then

$$\Delta_n = x_1 x_2 \dots x_n \left( 1 + \frac{a_1}{x_1} + \frac{a_2}{x_2} + \dots + \frac{a_{n-1}}{x_{n-1}} \right) \\ + a_n x_1 x_2 \dots x_{n-1} = x_1 x_2 \dots x_n \left( 1 + \frac{a_1}{x_1} + \frac{a_2}{x_2} + \dots + \frac{a_n}{x_n} \right).$$

$$213 \quad a_0 x_1 x_2 \dots x_n + a_1 y_1 x_2 \dots x_n + a_2 y_1 y_2 x_3 \dots x_n + \dots + \\ + a_n y_1 y_2 \dots y_n.$$

$$214 \quad -a_1 a_2 \dots a_n \left( \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right).$$

$$215 \quad n!(a_0 x^n + a_1 x^{n-1} + \dots + a_n).$$

$$216 \quad a_1 a_2 \dots a_{n-1} - a_1 a_2 \dots a_{n-2} + \dots + (-1)^n a_1 + (-1)^{n+1}.$$

217 Solution. Expand by minors of the first column:

$$\Delta_n = (\alpha + \beta) \Delta_{n-1} - \alpha \beta \Delta_{n-2}. \quad \text{We check} \quad \Delta_2 = \frac{\alpha^3 - \beta^3}{\alpha - \beta};$$

$$\Delta_3 = \frac{\alpha^4 - \beta^4}{\alpha - \beta}, \quad \text{and suppose} \quad \Delta_{n-2} = \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta};$$

$$\Delta_{n-1} = \frac{\alpha^n - \beta^n}{\alpha - \beta}. \quad \text{Then}$$

$$\Delta_n = (\alpha + \beta) \frac{\alpha^n - \beta^n}{\alpha - \beta} - \alpha \beta \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}.$$

Second Solution.

Write  $\Delta_n$  as  $d_n + \delta_n$ , where

$$d_n = \det \begin{bmatrix} \alpha & \alpha\beta & 0 & \dots & 0 & 0 \\ 1 & \alpha + \beta & \alpha\beta & \dots & 0 & 0 \\ 0 & 1 & \alpha + \beta & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 & \alpha + \beta \end{bmatrix},$$

$$\delta_n = \det \begin{bmatrix} \beta & 0 & 0 & \dots & 0 & 0 \\ 1 & \alpha + \beta & \alpha\beta & \dots & 0 & 0 \\ 0 & 1 & \alpha + \beta & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 & \alpha + \beta \end{bmatrix}.$$

We treat these matrices as follows. Factor  $\alpha$  from the first row of the first matrix; subtract the first row from the second. We obtain the result  $d_n = \alpha d_{n-1}$ . From the obvious relation  $d_2 = \alpha^2$ , we see that  $d_n = \alpha^n$  (this is really established by induction).

Now we expand the second determinant by minors of the elements of the first row and obtain the relation

$$\delta_n = \beta \Delta_{n-1}.$$

We now notice that  $\Delta_n = \alpha^n + \beta \Delta_{n-1}$ . Next we check the relation  $\Delta_2 = \frac{\alpha^2 - \beta^2}{\alpha - \beta}$ , and use the inductive hypothesis  $\Delta_{n-1} = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ . It follows easily that

$$\Delta_n = \alpha^n + \beta \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}.$$

$$218 \quad n + 1.$$

$$219 \quad \sin(n + 1)\theta / \sin \theta.$$

$$220 \quad \cos n\theta.$$

$$221 \quad x^n - C_1^{n-1} x^{n-2} + C_2^{n-2} x^{n-4} - \dots \quad \text{Cf. problem 53.}$$

$$222 \quad x_1 y_n \prod_{i=1}^n (x_{i+1} y_i - x_i y_{i+1}).$$

$$223 \quad a_1 a_2 \dots a_n \left( 1 + \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right).$$

$$224 \quad (-1)^{\frac{n(n-1)}{2}} a_1 a_2 \dots a_n \left( 1 + \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right).$$

$$225 \quad x(a_1 - x) \dots (a_n - x) \left( \frac{1}{x} + \frac{1}{a_1 - x} + \dots + \frac{1}{a_n - x} \right).$$

$$226 \quad (x_1 - a_1)(x_2 - a_2) \dots (x_n - a_n) \left( 1 + \frac{a_1}{x_1 - a_1} + \dots + \frac{a_n}{x_n - a_n} \right).$$

$$227 \quad \prod_{i=1}^n (x_i - a_i b_i) \left( 1 + \sum_{i=1}^n \frac{a_i b_i}{x_i - a_i b_i} \right).$$

$$228 \quad (-1)^n m^n \left( 1 - \sum_{i=1}^n \frac{x_i}{m} \right).$$

$$229 \quad x_1 = x_2 = \dots = x_{n-1} = 0; \quad x_n = \sum_{i=1}^n \frac{a_i}{a_i}.$$

$$230 \quad (a^2 - b^2)^n. \quad 231. \quad a(a+b) \dots [a + (n-1)b] \left( \frac{1}{a} + \frac{1}{a+b} + \dots + \frac{1}{a+(n-1)b} \right).$$

$$232 \quad x^{n-1} \prod_{i=1}^n (x - 2a_i) \left( x + \sum_{i=1}^n \frac{a_i^2}{x - 2a_i} \right).$$

$$233 \quad x^{n-1} \prod_{i=1}^n (x - 2a_i) \left( x + \sum_{i=1}^n \frac{a_i^2}{x - 2a_i} \right).$$

$$234 \quad 1 - b_1 + b_1 b_2 - b_1 b_2 b_3 + \dots + (-1)^n b_1 b_2 \dots b_n.$$

$$235 \quad (-1)^{n-1} (b_1 a_2 a_3 \dots a_n + b_1 b_2 a_3 \dots a_n + \dots + b_1 b_2 \dots b_{n-1} a_n).$$

$$236 \quad (-1)^{n-1} x^{n-2}. \quad 237 \quad (-1)^n [(x-1)^n - x^n].$$

$$238 \quad a_0 x^n \prod_{i=1}^n (b_i - a_i). \quad 240 \quad 1. \quad 241 \quad 1. \quad 242 \quad 1.$$

$$243 \quad \frac{C_{n+1}^{m+n} \cdot C_{n+1}^{m+n-1} \dots C_{n+1}^{m+n-k+1}}{C_{n+1}^{k+n} C_{n+1}^{k+n-1} \dots C_{n+1}^{n+1}}.$$

$$244 \quad (-1)^{\frac{m(m+1)}{2}}.$$

$$245 \quad (x - 1)^n$$

$$246 \quad (n - 1)! (n - 2)! \cdots 1! (x - 1)^n.$$

$$247 \quad \alpha^n.$$

248 The hints lead to

$$\Delta_n = (x - z)\Delta_{n-1} + z(x - y)^{n-1},$$

$$\Delta_n = (x - y)\Delta_{n-1} + y(x - z)^{n-1}.$$

Thus by elimination:

$$\Delta_n = \frac{z(x - y)^n - y(x - z)^n}{z - y}.$$

$$249 \quad (-1)^{\frac{n(n+1)}{2}} \frac{ab(b^{n-1} - a^{n-1})}{a - b}.$$

$$250 \quad \frac{xf(y) - yf(x)}{x - y}, \text{ where } f(x) = \prod_{k=1}^n (a_k - x).$$

$$251 \quad \frac{f(a) - f(b)}{a - b}, \text{ where } f(x) = \prod_{k=1}^n (c_k - x).$$

$$252 \quad (\alpha - \beta)^{n-2} [\lambda\alpha + (n - 2)\lambda\beta - (n - 1)ab].$$

$$253 \quad (-1)^{\frac{n(n-1)}{2}} \frac{n^{n-1}(n+1)}{2}.$$

$$254 \quad (-1)^{\frac{n(n-1)}{2}} (nh)^{n-1} \left[ a + \frac{h(n-1)}{2} \right].$$

$$255 \quad (1 - x^n)^{n-1}.$$

256 If we add the last three columns to the first we note that the determinant is a rational function of  $a$  that has the quantity  $a + b + c + d$  as a factor.

We can show that the value of the determinant is divisible by  $a + b - c - d$  as follows: To the first column add the second and subtract the

third and fourth. By a variation of this method, we see that the value of the determinant is also divisible by  $a - b + c - d$ ,  $a - b - c + d$ . Thus the determinant has the form

$\lambda(a + b + c + d)(a + b - c - d)(a - b + c - d)(a - b - c + d)$ . Since the determinant is clearly a fourth-degree polynomial in  $a$ ,  $\lambda$  must be a constant. The coefficient of  $a^4$  is 1; therefore  $\lambda = 1$ .

$$257 \quad (a+b+c+d+e+f+g+h)(a+b+c+d-e-f-g-h) \\ \times (a+b-c-d+e+f-g-h)(a+b-c-d-e-f+g+h) \\ \times (a-b+c-d+e-f+g-h)(a-b+c-d-e+f-g+h) \\ \times (a-b-c+d+e-f-g+h)(a-b-c+d-e+f+g-h).$$

$$258 \quad (x+a_1+a_2+\dots+a_n)(x-a_1)(x-a_2)\dots(x-a_n).$$

$$259 \quad 2^{\frac{n(n-1)}{2}} \prod_{1 \leq i < k \leq n} \sin \frac{\varphi_i + \varphi_k}{2} \prod_{1 \leq i < k \leq n} \sin \frac{\varphi_k - \varphi_i}{2}.$$

$$260 \quad 2^{\frac{n(n-1)}{2}} \prod_{n \geq i > k \geq 1} \cos \frac{\varphi_i + \varphi_k}{2} \prod_{n \geq i > k \geq 1} \sin \frac{\varphi_i - \varphi_k}{2}.$$

$$261 \quad 1! \cdot 2! \cdot \dots \cdot n!.$$

$$262 \quad \prod_{n+1 \geq k > i \geq 1} (a_i - a_k).$$

$$263 \quad (-1)^n 1! \cdot 2! \cdot \dots \cdot n!$$

$$264 \quad (-1)^{n-1} \prod_{i=1}^n a_i \prod_{n \geq i > k \geq 1} (a_i - a_k) \left( \sum_{i=1}^n \frac{w_i}{a_i f'(a_i)} \right),$$

where  $f(x) = (x - a_1)(x - a_2) \dots (x - a_n)$ .

- 265  $\prod_{n \geq i > k \geq 1} (x_i - x_k).$
- 266  $2^{\frac{n(n-1)}{2}} \prod_{n \geq i > k \geq 1} \cos \frac{\varphi_i + \varphi_k}{2} \prod_{n \geq i > k \geq 1} \sin \frac{\varphi_i - \varphi_k}{2}.$
- 267  $\prod_{n \geq i > k \geq 1} (x_i - x_k).$
- 268  $2^{\frac{n(n-1)}{2}} a_{01} a_{02} \dots a_{0n-1} \prod_{n \geq i > k \geq 1} \sin \frac{\varphi_i + \varphi_k}{2} \times$   
 $\times \prod_{n \geq i > k \geq 1} \sin \frac{\varphi_k - \varphi_i}{2}.$
- 269  $\frac{1}{1! 2! \dots (n-1)!} \prod_{n \geq i > k \geq 1} (x_i - x_k).$
- 271  $1! 3! 5! \dots (2n-1)!$  272  $\prod_{i=1}^n \frac{x_i}{x_i - 1} \prod_{n \geq i > k \geq 1} (x_i - x_k).$
- 273  $\prod_{n+1 \geq k > i \geq 1} (b_k a_i - a_k b_i).$  274  $\prod_{1 \leq i < k \leq n} \sin (\alpha_i - \alpha_k).$
- 275  $\prod_{1 \leq i < k \leq n+1} (a_i - a_k) (a_i a_k - 1).$
- 276  $2^{(n-1)^2} \prod_{n-1 \geq i > k \geq 0} \sin \frac{\varphi_i + \varphi_k}{2} \prod_{n-1 \geq i > k \geq 0} \sin \frac{\varphi_k - \varphi_i}{2}.$
- 277  $2^n (n+1) \sin \alpha_0 \sin \alpha_1 \dots$   
 $\dots \sin \alpha_n \prod_{n \geq i > k \geq 0} \sin \frac{\alpha_i + \alpha_k}{2} \prod_{n \geq i > k \geq 0} \sin \frac{\alpha_i - \alpha_k}{2}.$
- 278  $[x_1 x_2 \dots x_n - (x_1 - 1) \dots (x_n - 1)] \prod_{n \geq i > k \geq 1} (x_i - x_k).$
- 279  $x_1 x_2 \dots x_n \left( \frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} \right) \prod_{n \geq i > k \geq 1} (x_i - x_k).$
- 280  $(x_1 + x_2 + \dots + x_n) \prod_{n \geq i > k \geq 1} (x_i - x_k).$

$$281 \quad \sigma_{n-s} = \sum_{n \geq i > k \geq 1} \prod (x_i - x_k), \text{ where } \sigma_p \text{ denotes the}$$

sum of all possible products of  $x_1, x_2, \dots, x_n$ ,  
taken  $p$  at a time.

$$282 \quad [2x_1x_2 \dots x_n - (x_1 - 1)(x_2 - 1) \dots (x_n - 1)] \times \\ \times \sum_{n \geq i > k \geq 1} (x_i - x_k).$$

$$283 \quad x^2(x^2 - 1)^4.$$

$$284 \quad 2x^3y(x - y)^6.$$

$$285 \quad 1! 2! 3! \dots (n-1)! x^{\frac{n(n-1)}{2}} (y - x)^n.$$

$$286 \quad 1! 2! 3! \dots (k-1)! x^{\frac{k(k-1)}{2}} (y_1 - x)^k (y_2 - x)^k \dots \\ \dots (y_{n-k} - x)^k \prod_{n-k \geq i > j \geq 1} (y_i - y_j).$$

$$287 \quad (y - x)^{k(n-k)}.$$

$$288 \quad \begin{aligned} & \text{b) } 9; \text{ c) } 5; \text{ e) } 128; \text{ f) } (a_1a_2 - b_1b_2)(c_1c_2 - d_1d_2); \\ & \text{g) } (x_3 - x_2)^2(x_3 - x_1)^2(x_2 - x_1)^2; \\ & \text{h) } (k^2 - a^2)(\alpha - \beta)^{n-1} [\alpha + (n-1)\beta]; \\ & \text{k) } (x_4 - x_3)[(x_3 - x_2)(x_4 - x_2) - 2(x_3 - x_1)(x_4 - x_1)]; \\ & \text{m) } 27(a+2)^3(a-1)^6[3(a+2)^2 - 4x^2][3(a-1)^2 - 4x^2]^2. \end{aligned}$$

Note: This problem is a special case of 537.



289

$$\begin{aligned} \text{a) } & \begin{vmatrix} -5 & -2 \\ -8 & 4 \end{vmatrix}; & \text{b) } & \begin{vmatrix} 2 & 8 & 17 \\ 11 & -6 & 5 \\ 3 & 8 & -3 \end{vmatrix}; \\ \text{c) } & \begin{vmatrix} 7 & 5 & -3 & 3 \\ -1 & 5 & -3 & 3 \\ -4 & -4 & -5 & 4 \\ -4 & -4 & 0 & 3 \end{vmatrix}. \end{aligned}$$

290 a) 24; b) 18;

c)  $(a + b + c + d)(a + b - c - d)(a - b + c - d)(a - b - c + d)$ .

291 a) 16; b) 280; c)  $(a^2 + b^2 + c^2 + d^2)^2$ .

292  $D \prod_{n \geq i > k \geq 1} (x_i - x_k)$ .

293 a)  $C_1^n C_2^n \cdots C_n^n \prod_{n \geq i > k \geq 0} (a_i - a_k)(b_k - b_i)$ .

b)  $\prod_{n \geq i > k \geq 1} (\alpha_i - \alpha_k)(\beta_i - \beta_k)$ .

294 0, if  $n \geq 2$ .

295  $\prod_{i=1}^n (x - x_i) \prod_{n \geq i > k \geq 1} (x_i - x_k)^2$ .

296  $-(a^2 + b^2 + c^2 + d^2 + l^2 + m^2 + n^2 + p^2)^4$ .

297  $4 \sin^4 \varphi$ .

298  $4 \sin^4 \varphi$ .

- 299 Let the value of the determinant be  $\Delta_n$ . By squaring the matrix involved, we see that  $\det \Delta = n^{n/2}$ . On the other hand,

$$\Delta = \prod_{n-1 \geq k > s \geq 0} (\epsilon^k - \epsilon^s).$$

Set  $\epsilon_1 = \cos \frac{\pi}{n} + i \sin \frac{\pi}{n}$ . Then  $\epsilon = \epsilon_1^2$ ,

$$\begin{aligned} \Delta &= \prod_{n-1 \geq k > s \geq 0} (\epsilon^k - \epsilon^s) = \prod \epsilon_1^{k+s} \prod (\epsilon_1^{k-s} - \epsilon_1^{-k+s}) = \\ &= \prod \epsilon_1^{k+s} \cdot i^{\frac{n(n-1)}{2}} \prod 2 \sin \frac{(k-s)\pi}{n}. \quad \text{In fact} \\ \sin \frac{(k-s)\pi}{n} &> 0, \text{ for all } k, s. \text{ Thus} \end{aligned}$$

$$n^{\frac{n}{2}} = |\Delta| = \left| \prod 2 \sin \frac{(k-s)\pi}{n} \right| = \prod 2 \sin \frac{(k-s)\pi}{n}.$$

and

$$\begin{aligned} \Delta &= n^{\frac{n}{2}} i^{\frac{n(n-1)}{2}} \prod_{n-1 \geq k > s \geq 0} \epsilon_1^{k+s} = n^{\frac{n}{2}} i^{\frac{n(n-1)}{2}} \epsilon_1^{\frac{n(n-1)^2}{2}} = \\ &= n^{\frac{n}{2}} i^{\frac{n(n-1)}{2} + (n-1)^2} = n^{\frac{n}{2}} i^{\frac{(n-1)(n-2)}{2}}. \end{aligned}$$

300  $\prod_{k=0}^{n-1} (a_0 + a_1 \epsilon_k + a_2 \epsilon_k^2 + \dots + a_{n-1} \epsilon_k^{n-1})$ , where

$$\epsilon_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}.$$

301  $x^4 - y^4 + z^4 - u^4 + 4xy^2z + 4xzu^2 - 4x^2yu - 4yz^2u - 2x^2z^2 + 2y^2u^2.$

303  $2^{n-1}$ , if  $n$  is odd; 0, if  $n$  is even.

304  $(-1)^n \frac{[(n+1)a^n - 1]^n - n^n a^{n(n+1)}}{(1-a^n)^2}.$

$$305 \quad (-1)^{n-1} (n-1) \prod_{k=0}^{n-1} (a_1 + a_2 \varepsilon_k + \dots + a_n \varepsilon_k^{n-1}) \quad , \text{ where}$$

$$\varepsilon_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}.$$

$$306 \quad \varphi_0(t) \varphi_1(t) \dots \varphi_{n-1}(t), \quad \text{where} \quad \varphi_k(t) = \frac{(t + \varepsilon_k)^n - 1}{t};$$

$$\varepsilon_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}.$$

By problem 102, the result can be written

$$\prod_{k=1}^{n-1} [t^n - (\varepsilon_k - 1)^n].$$

$$307 \quad (-2)^{n-1} (n-2p) \quad , \text{ if } (n, p) = 1; \quad 0, \text{ if } (n, p) \neq 1.$$

$$308 \quad 2^{n-2} \left( \cos^n \frac{\pi}{n} - 1 \right).$$

$$309 \quad 2^{n-2} \sin^{n-2} \frac{n\theta}{2} \left[ \sin^n \frac{(n+2)\theta}{2} - \sin^n \frac{n\theta}{2} \right].$$

$$310 \quad (-1)^n 2^{n-2} \sin^{n-2} \frac{nh}{2} \left[ \cos^n \left( a + \frac{nh}{2} \right) - \cos^n \left( a + \frac{(n-2)h}{2} \right) \right].$$

$$311 \quad (-1)^{n-1} \frac{(n+1)(2n+1)}{12} n^{n-2} [(n+2)^n - n^n].$$

$$313 \quad \prod_{k=0}^{n-1} (a_1 + a_2 \varepsilon_k + a_3 \varepsilon_k^2 + \dots + a_n \varepsilon_k^{n-1}),$$

$$\text{where} \quad \varepsilon_k = \cos \frac{(2k+1)\pi}{n} + i \sin \frac{(2k+1)\pi}{n}.$$

$$315 \quad \prod_{i=1}^n (a_1 + a_2 \rho_i + a_3 \rho_i^2 + \dots + a_n \rho_i^{n-1}),$$

where  $\rho_1, \rho_2, \dots, \rho_n$  are  $n$ -th roots of  $\mu$ .

318 As for 223: If we add 1 to each element of the matrix

$$\begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & a_n \end{bmatrix}$$

we obtain the matrix of 223.

$$\text{Thus } \Delta = a_1 a_2 \dots a_n + \sum_{k=1}^n \sum_{i=1}^n A_{ik};$$

$$\sum_{k=1}^n \sum_{i=1}^n A_{ik} = a_1 a_2 \dots a_n \left( \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right).$$

As for 250: the determinant  $\Delta$  satisfies:

$$\Delta = (a_1 - x)(a_2 - x) \dots (a_n - x) + x \sum A_{ik};$$

$$\Delta = (a_1 - y)(a_2 - y) \dots (a_n - y) + y \sum A_{ik},$$

where  $\sum A_{ik}$  is the sum of the algebraic cofactors of all the elements of  $\Delta$ . Using these equations the value of  $\Delta$  is easily determined.

$$323 \quad \prod_{1 \leq i < k \leq n} (a_i - a_k) \prod_{1 \leq i < k \leq n} (b_i - b_k) \cdot \frac{1}{\prod_{i=1}^n f(a_i)},$$

where  $f(x) = (x + b_1) \dots (x + b_n)$ .

$$325 \quad \frac{[c + \sqrt{c^2 - 4ab}]^{n+1} - [c - \sqrt{c^2 - 4ab}]^{n+1}}{2^{n+1} \sqrt{c^2 - 4ab}}.$$

$$326 \quad \frac{[p + \sqrt{p^2 - 4q}]^n + [p - \sqrt{p^2 - 4q}]^n}{2^n}.$$

- 327  $x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n$  , where  $a_k$  is the sum of all the  $k$ -th order minors of the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix},$$

that are contained in the  $n - k$  rows with indices  $\alpha_1, \alpha_2, \dots, \alpha_{n-k}$  , and columns with the same indices.

328  $(n+1)^{n-1}$ .

329  $(x-n)^{n+1}$ .

330  $(x^2 - 1^2)(x^2 - 3^2) \dots [x^2 - (2m-1)^2]$  , if  $n = 2m$ ;  
 $x(x^2 - 2^2)(x^2 - 4^2) \dots (x^2 - 4m^2)$  , if  $n = 2m + 1$ .

331  $(x + na - n)[x + (n-2)a - n + 1] \times$   
 $\times [x + (n-4)a - n + 2] \dots (x - na)$ .

332  $(-1)^{\frac{n(n-1)}{2}} [(n-1)!]^n$ .

333  $\frac{[1!2! \dots (n-1)!]^3}{n!(n+1)! \dots (2n-1)!}$ .

334  $\frac{\Delta(a_1, a_2, \dots, a_n) \Delta(b_1, b_2, \dots, b_n)}{\Delta(1, 2, \dots, n)}$  , where  $\Delta$  is the Vandermonde determinant.

CHAPTER III - SOLUTIONS  
SYSTEMS OF LINEAR EQUATIONS

- 335  $x_1 = 3; x_2 = x_3 = 1.$  336  $x_1 = 1; x_2 = 2; x_3 = -2.$   
 337  $x_1 = 2; x_2 = -2; x_3 = 3.$  338  $x_1 = 3; x_2 = 4; x_3 = 5.$   
 339  $x_1 = x_2 = -1; x_3 = 0; x_4 = 1.$   
 340  $x_1 = 1; x_2 = 2; x_3 = -1; x_4 = -2.$   
 341  $x_1 = -2; x_2 = 2; x_3 = -3; x_4 = 3.$   
 342  $x_1 = 1; x_2 = 2; x_3 = 1; x_4 = -1.$   
 343  $x_1 = 2; x_2 = x_3 = x_4 = 0.$  344  $x_1 = x_2 = x_3 = x_4 = 0.$   
 345  $x_1 = 1; x_2 = -1; x_3 = 0; x_4 = 2.$   
 346  $x_1 = x_2 = x_3 = x_4 = x_5 = 0.$  347  $x_1 = x_2 = x_3 = x_4 = 0.$   
 348  $x_1 = 1; x_2 = -1; x_3 = 1; x_4 = -1; x_5 = 1.$   
 349  $x_1 = x_2 = x_3 = x_4 = x_5 = 0.$   
 350  $x_1 = 1; x_2 = -1; x_3 = 1; x_4 = -1; x_5 = 1.$   
 351  $x_1 = 0; x_2 = 2; x_3 = -2; x_4 = 0; x_5 = 3.$   
 352  $x_1 = 2; x_2 = 0; x_3 = -2; x_4 = -2; x_5 = 1.$

353 Since the system has a non-trivial solution, the determinant of the matrix of coefficients must be zero.

354 The determinant of the matrix of coefficients is  $-(a^2 + b^2 + c^2 + d^2)^2.$

355 
$$x_i = \frac{\alpha \sum_{k=1}^n a_k - a_i [(n-1)\alpha + \beta]}{(\alpha - \beta) [(n-1)\alpha + \beta]}.$$

$$356 \quad x_i = - \frac{f(\beta_i)}{\varphi'(\beta_i)}, \quad \text{where} \quad \begin{aligned} f(x) &= (x - b_1)(x - b_2) \dots (x - b_n). \\ \varphi(x) &= (x - \beta_1)(x - \beta_2) \dots (x - \beta_n). \end{aligned}$$

$$357 \quad x_i = \frac{f(t)}{(t - \alpha_i)f'(\alpha_i)}, \quad \text{where} \quad f(x) = (x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n).$$

$$358 \quad x_s = \sum_{i=1}^n \frac{(-1)^{n+s} u_i}{f'(\alpha_i)} \varphi_{s,i}, \quad \text{where} \quad \begin{aligned} f(x) &= (x - \alpha_1) \dots (x - \alpha_n); \\ \varphi_{s,i} &= \sum \alpha_{t_1} \alpha_{t_2} \dots \alpha_{t_{n-s}}, \end{aligned}$$

where the sum is taken over all subsets

$t_1, t_2, \dots, t_{n-s}$  of  $1, 2, \dots, i-1, i+1, \dots, n$ .

$$359 \quad x_s = \sum_{i=1}^n \frac{(-1)^{n+i} u_i}{f'(\alpha_i)} \varphi_{i,s}, \quad \text{where} \quad \begin{aligned} f(x) &= (x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n); \\ \varphi_{i,s} &= \sum \alpha_{t_1} \alpha_{t_2} \dots \alpha_{t_{n-i}}, \end{aligned}$$

where the sum is taken over

all subsets  $t_1, t_2, \dots, t_{n-i}$  of  $1, 2, \dots, s-1,$

$s+1, \dots, n$ .

$$360 \quad x_i = \frac{(-1)^n a_{n-i}}{n!}, \quad \text{where} \quad \begin{aligned} x^n + a_1 x^{n-1} + \dots + a_n \\ = (x-1)(x-2) \dots (x-n). \end{aligned}$$

$$361 \quad C_k^m \cdot C_k^n.$$

365 a) Either unchanged or increased by unity.

b) Either unchanged or increased by one or two units.

$$366 \quad 2. \qquad \qquad \qquad 370 \quad 3.$$

$$367 \quad 3. \qquad \qquad \qquad 371 \quad 3.$$

$$368 \quad 2. \qquad \qquad \qquad 372 \quad 4.$$

$$369 \quad 2. \qquad \qquad \qquad 373 \quad 3.$$

- |             |  |     |    |
|-------------|--|-----|----|
| 374         | 2.   | 377 | 6. |
| 375         | 3.   | 378 | 5. |
| 376         | 5.   | 379 | 3. |
|             |  | 380 | 4. |
| 383         | The forms are independent.   |     |    |
| 384         | $2y_1 - y_2 - y_3 = 0$ .   |     |    |
| 385         | $y_1 + 3y_2 - y_3 = 0$ ; $2y_1 - y_2 - y_4 = 0$ .                    |     |    |
| 386         | The forms are independent.   |     |    |
| 387         | $y_1 + y_2 - y_3 - y_4 = 0$ .  |     |    |
| 388         | $y_1 - y_2 + y_3 = 0$ ; $5y_1 - 4y_2 + y_4 = 0$ .                    |     |    |
| 389,<br>390 | The forms are independent.   |     |    |
| 391         | $y_1 + y_2 - y_3 - y_4 = 0$ .  |     |    |
| 392         | $2y_1 - y_2 - y_3 = 0$ .   |     |    |
| 393         | $3y_1 - y_2 - y_3 = 0$ ; $y_1 - y_2 - y_4 = 0$ .                     |     |    |
| 394         | The forms are independent.   |     |    |
| 395         | $y_1 - y_2 - y_3 - y_4 = 0$ .  |     |    |
| 396         | $3y_1 - 2y_2 - y_3 + y_4 = 0$ ; $y_1 - y_2 + 2y_3 - y_5 = 0$ .       |     |    |
| 397         | $\lambda = 10$ ; $3y_1 + 2y_2 - 5y_3 - y_4 = 0$ .                    |     |    |
| 398         | $x_3 = 2x_2 - x_1$ ; $x_4 = 1$ .                                     |     |    |
| 399         | $\lambda = 5$ .  |     |    |
| 400         | No solution.   |     |    |
| 401         | $x_1 = 1$ ; $x_2 = 2$ ; $x_3 = -2$ .                                 |     |    |
| 402         | $x_1 = 1$ ; $x_2 = 2$ ; $x_3 = 1$ .                                  |     |    |
| 403         | $x_1 = -\frac{11x_3}{7}$ ; $x_2 = -\frac{x_3}{7}$ .                  |     |    |
| 404         | No solution.   |     |    |
| 405         | $x_1 = 0$ ; $x_2 = 2$ ; $x_3 = \frac{5}{3}$ ; $x_4 = -\frac{4}{3}$ . |     |    |



$$406 \quad x_1 = -8; \quad x_2 = 3 + x_4; \quad x_3 = 6 + 2x_4.$$

$$407 \quad x_1 = 2; \quad x_2 = x_3 = x_4 = 1.$$

$$408 \quad x_1 = x_2 = x_3 = x_4 = 0.$$

$$409 \quad x_1 = \frac{3x_3 - 13x_4}{17}; \quad x_2 = \frac{19x_3 - 20x_4}{17}.$$

$$410 \quad x_1 = \frac{7}{6}x_5 - x_3; \quad x_2 = \frac{5}{6}x_5 + x_3; \quad x_4 = \frac{x_5}{3}.$$

$$411 \quad x_1 = -16 + x_3 + x_4 + 5x_5; \quad x_2 = 23 - 2x_3 - 2x_4 - 6x_5.$$

$$412 \quad x_1 = \frac{-4x_4 + 7x_5}{8}; \quad x_2 = \frac{-4x_4 + 5x_5}{8}; \quad x_3 = \frac{4x_4 - 5x_5}{8}.$$

$$413 \quad x_1 = x_2 = x_3 = 0; \quad x_4 = x_5.$$

$$414 \quad x_1 = \frac{1 + x_5}{3}; \quad x_2 = \frac{1 + 3x_3 + 3x_4 - 5x_5}{3}.$$

$$415 \quad x_1 = \frac{2 + x_5}{3}; \quad x_2 = \frac{1 + 3x_3 - 3x_4 + 5x_5}{6}.$$

$$416, \quad \text{No solution.}$$

$$417 \quad x_1 = -\frac{x_5}{2}; \quad x_2 = -1 - \frac{x_5}{2}; \quad x_3 = 0; \quad x_4 = -1 - \frac{x_5}{2}.$$

$$419 \quad x_1 = \frac{1 + 5x_4}{6}; \quad x_2 = \frac{1 - 7x_4}{6}; \quad x_3 = \frac{1 + 5x_4}{6}.$$

$$420 \quad \text{No solution.}$$

$$421 \quad x = \frac{b^2 + c^2 - a^2}{2bc}; \quad y = \frac{a^2 + c^2 - b^2}{2ac}; \quad z = \frac{a^2 + b^2 - c^2}{2ab}.$$

$$422 \quad \text{If } (\lambda - 1)(\lambda + 2) \neq 0, \quad x = -\frac{\lambda + 1}{\lambda + 2}; \quad y = \frac{1}{\lambda + 2};$$

$$z = \frac{(\lambda + 1)^2}{\lambda + 2}.$$

If  $\lambda = 1$ , the solution of the system depends on two independent parameters.

If  $\lambda = -2$ , there is no solution.

423 If  $(\lambda - 1)(\lambda + 3) \neq 0$ , then  $x = -\frac{\lambda^2 + 2\lambda + 2}{\lambda + 3}$ ;  
 $y = -\frac{\lambda^2 + \lambda - 1}{\lambda + 3}$ ;  $z = \frac{2\lambda + 1}{\lambda + 3}$ ;  $t = \frac{\lambda^3 + 3\lambda^2 + 2\lambda + 1}{\lambda + 3}$ .

If  $\lambda = 1$ , the solution depends on three independent parameters.

If  $\lambda = -3$ , there is no solution.

424 If  $a, b, c$  are all distinct,

$$x = abc; \quad y = -(ab + ac + bc); \quad z = a + b + c.$$

If exactly two of  $a, b, c$  are equal the solution depends on one parameter.

If  $a = b = c$ , the solution set is expressed linearly in terms of two independent parameters.

425 If  $a, b, c$  are all distinct, then

$$x = \frac{(b-d)(c-d)}{(b-a)(c-a)}; \quad y = \frac{(a-d)(c-d)}{(a-b)(c-b)}; \quad z = \frac{(a-d)(b-d)}{(a-c)(b-c)}.$$

If  $a = b$ ;  $a \neq c$ ;  $d = a$  or  $d = c$ , the solution set is a one-parameter set.

If  $b = c$ ;  $a \neq b$ ;  $d = a$  or  $d = b$ , the solution set is a one-parameter set.

If  $a = c$ ;  $a \neq b$ ;  $d = a$  or  $d = b$ , the solution set is one-parameter set.

If  $a = b = c = d$ , the solution set is a two-parameter set.

In all remaining cases, the system has no solution.

- 426 If  $b(a - 1) \neq 0$ , then

$$x = \frac{2b-1}{b(a-1)}; y = \frac{1}{b}; z = \frac{2ab-4b+1}{b(a-1)}.$$

If  $a = 1$ ;  $b = 1/2$ , the solution set is a one-parameter set.

In any other case, there is no solution.

- 427 If  $b(a - 1)(a + 2) \neq 0$ , then  $x = z = \frac{a-b}{(a-1)(a+2)}$ ;  
 $y = \frac{ab+b-2}{b(a-1)(a+2)}.$

If  $a = -2$ ;  $b = -2$ , the solution depends on one parameter.

If  $a = 1$ ;  $b = 1$ , the solution depends on two parameters.

No solution in the remaining cases.

- 428 If  $(\alpha - 1)(\alpha + 2) \neq 0$ , then  $x = \frac{m\alpha + m - n - p}{(\alpha + 2)(\alpha - 1)}$ ;  
 $y = \frac{n\alpha + n - m - p}{(\alpha + 2)(\alpha - 1)}$ ;  $z = \frac{p\alpha + p - m - n}{(\alpha + 2)(\alpha - 1)}.$

If  $\alpha = -2$  and  $m + n + p = 0$ , the solution depends on one parameter.

If  $\alpha = 1$  and  $m = n = p$ , the solution depends on two parameters.

No solution in the remaining cases.

- 429 If  $a(a - b) \neq 0$ , then

$$x = \frac{a^2(b-1)}{b-a}; y = \frac{b(a^2-1)}{a(a-b)}; z = \frac{a-1}{a(b-a)}.$$

If  $a = b = 1$ , the solution depends on two parameters.

No solution in the remaining cases.

- 430  $\Delta = \lambda^2(\lambda - 1)$ . For  $\lambda = 0$ ;  $\lambda = 1$  the system is inconsistent (has no solution).
- 431  $\Delta = -2\lambda$ . If  $\lambda \neq 0$ , then  $x = 1 - \lambda$ ,  $y = \lambda$ ,  $z = 0$ .  
If  $\lambda = 0$ , then  $x = 1$ ,  $z = 0$ , and  $y$  is arbitrary.
- 432  $\Delta = (k - 1)^2(k + 1)$ . If  $k = 1$ , the solution set is a one-parameter set. If  $k = -1$ , the system is inconsistent.
- 433  $\Delta = a(b - 1)(b + 1)$ .  
If  $a = 0$ ,  $b = 5$ ; then  $y = -1/3$ ,  $z = 4/3$ , with  $x$  arbitrary.  
If  $a = 0$ ,  $b \neq 1$ ,  $b \neq 5$ , the system is inconsistent.  
If  $b = 1$ , then  $z = 0$ ,  $y = 1 - ax$ ,  $x$  arbitrary.  
If  $b = -1$ , the system is inconsistent.
- 434 a)  $\Delta = -m(m + 2)$ . For  $m = 0$ ,  $m = -2$  the system is inconsistent.  
b)  $\Delta = m(m^2 - 1)$ . If  $m = 0$  or  $m = 1$  the system is inconsistent. If  $m = -1$  the solution set is a one parameter set.  
c)  $\Delta = \lambda(\lambda - 1)(\lambda + 1)$ . If  $\lambda = 1$  or  $\lambda = -1$  the system is inconsistent. If  $\lambda = 0$  the solution set is a one parameter set.
- 435 a)  $\Delta = 3(c + 1)(c - 1)^2$ . If  $c = -1$ , the system is inconsistent. If  $c = 1$ , the solution set is a two-parameter set.  
b)  $\Delta = (\lambda - 1)(\lambda - 2)(\lambda - 3)$ . If  $\lambda = 2$  or  $\lambda = 3$ , the system is inconsistent. If  $\lambda = 1$ , the solution set is a one parameter set.

c)  $\Delta = d(d-1)(d+2)$ . If  $d = 1$  or  $d = -2$ , the system is inconsistent. If  $d = 0$  the solution set is a one parameter set.

d)  $\Delta = (a-1)^2(a+1)$ . If  $a = -1$ , the system is inconsistent. If  $a = 1$ , the solution set is a two parameter set.

$$436 \quad \det \begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} = 0.$$

437 If and only if

$$\det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0.$$

438

$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = 0.$$

439 Only if

$$\det \begin{bmatrix} x_0^2 + y_0^2 & x_0 & y_0 & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{bmatrix} = 0.$$

$$440 \quad (x-1)^2 + (y-1)^2 = 1.$$

$$441 \quad y^2 - y = 0.$$

$$442 \quad y = x^3 - 1.$$

443

$$\det \begin{bmatrix} y & x^n & x^{n-1} & \dots & x^2 & x & 1 \\ y_0 & x_0^n & x_0^{n-1} & \dots & x_0^2 & x_0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ y_n & x_n^n & x_n^{n-1} & \dots & x_n^2 & x_n & 1 \end{bmatrix} = 0.$$

444 If and only if

$$\det \begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix} = 0.$$

445  $x^2 + y^2 + z^2 - x - y - z = 0$ .

446 If and only if the rank of the matrix

$$\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ \cdot & \cdot & \cdot \\ x_n & y_n & 1 \end{pmatrix}$$

is less than three.

447 If and only if the rank of the matrix

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ \cdot & \cdot & \cdot \\ a_n & b_n & c_n \end{pmatrix}$$

is less than three.

448 In the same plane, if and only if the rank of the matrix

$$\begin{pmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ \cdot & \cdot & \cdot & \cdot \\ x_n & y_n & z_n & 1 \end{pmatrix}$$

is less than four; on a single line if and only if the rank of the same matrix is less than three.

449 All the planes pass through a single point, if the rank of the matrix

$$\begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ \cdot & \cdot & \cdot & \cdot \\ A_n & B_n & C_n & D_n \end{pmatrix}$$

is less than four; through a single line if the rank of the same matrix is less than three.

450

$$\det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1, n-1} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2, n-1} & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{n, n-1} & a_{nn} \end{bmatrix} = 0.$$

453 No.

454 For example,

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 \\ 5 & -6 & 0 & 0 & 1 \end{pmatrix}.$$

455 Yes.

456 Solution. Set

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{r1} & \alpha_{r2} & \cdots & \alpha_{rn} \end{pmatrix}; \quad B = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1r} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2r} \\ \cdot & \cdot & \cdot & \cdot \\ \lambda_{r1} & \lambda_{r2} & \cdots & \lambda_{rr} \end{pmatrix};$$

$$BA = \begin{pmatrix} \sum_{s=1}^r \lambda_{1s} \alpha_{s1} & \cdots & \sum_{s=1}^r \lambda_{1s} \alpha_{sn} \\ \cdot & \cdot & \cdot & \cdot \\ \sum_{s=1}^r \lambda_{rs} \alpha_{s1} & \cdots & \sum_{s=1}^r \lambda_{rs} \alpha_{sn} \end{pmatrix}.$$

It is immediately obvious that the rows of the matrix  $BA$  form solutions of the system. Moreover since  $\det B \neq 0$ , we have  $A = B^{-1}(BA)$ . This says that the solutions making up the rows of the matrix  $A$  are linear combinations of the solutions that make up the rows of the matrix  $BA$ .

457 Solution. Let

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{r1} & \alpha_{r2} & \cdots & \alpha_{rn} \end{pmatrix}; \quad C = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1n} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{r1} & \gamma_{r2} & \cdots & \gamma_{rn} \end{pmatrix}.$$

Since  $C$  is a set of fundamental solutions, we have  $\alpha_{11} = \lambda_{11}\gamma_{11} + \lambda_{12}\gamma_{21} + \cdots + \lambda_{1r}\gamma_{r1}$ , etc. Thus  $A = BC$ , where

$$B = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1r} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{r1} & \lambda_{r2} & \cdots & \lambda_{rr} \end{pmatrix}.$$

On the other hand  $A$  is also a set of fundamental solutions of system, and therefore  $\det B \neq 0$ .

459 For example

$$x_1 = c_1 + c_2 + 5c_3; \quad x_2 = -2c_1 - 2c_2 - 6c_3; \quad x_3 = c_1; \quad x_4 = c_2; \quad x_5 = c_3.$$

See the solution to problem 454.

460  $x_1 = 11c; \quad x_2 = c; \quad x_3 = -7c.$



461    № 408.  $x_1 = x_2 = x_3 = x_4 = 0$ .

№ 409.  $x_1 = 3c_1 + 13c_2$ ;  $x_2 = 19c_1 + 20c_2$ ;  $x_3 = 17c_1$ ;  $x_4 = -17c_2$ .

№ 410.  $x_1 = c_1 + 7c_2$ ;  $x_2 = -c_1 + 5c_2$ ;  $x_3 = -c_1$ ;  $x_4 = 2c_2$ ;  $x_5 = 6c_2$ .

№ 412.  $x_1 = c_1 + 7c_2$ ;  $x_2 = c_1 + 5c_2$ ;  $x_3 = -c_1 - 5c_2$ ;  $x_4 = -2c_1$ ;  
 $x_5 = 8c_2$ .

№ 413.  $x_1 = 0$ ;  $x_2 = 0$ ;  $x_3 = 0$ ;  $x_4 = c$ ;  $x_5 = c$ .

462     $x_1 = -16 + c_1 + c_2 + 5c_3$ ;

$x_2 = 23 - 2c_1 - 2c_2 - 6c_3$ ;

$x_3 = c_1$ ;  $x_4 = c_2$ ;  $x_5 = c_3$ .

463    № 406.  $x_1 = -8$ ;  $x_2 = 3 + c$ ;  $x_3 = 6 + 2c$ ;  $x_4 = c$ .

№ 414.  $x_1 = c_3$ ;  $x_2 = 2 + c_1 + c_2 - 5c_3$ ;  $x_3 = c_1$ ;  $x_4 = c_2$ ;  $x_5 = -1 + 3c_3$ .

№ 415.  $x_1 = 1 + 2c_3$ ;  $x_2 = 1 + c_1 - c_2 + 5c_3$ ;  $x_3 = 2c_1$ ;  $x_4 = 2c_2$ ;  
 $x_5 = 1 + 6c_3$ .

# CHAPTER IV - SOLUTIONS

## MATRICES

464 a)  $\begin{pmatrix} 3 & -1 \\ 5 & -1 \end{pmatrix}$ ; b)  $\begin{pmatrix} -9 & 13 \\ 15 & 4 \end{pmatrix}$ ; c)  $\begin{pmatrix} 6 & 2 & -1 \\ 6 & 1 & 1 \\ 8 & -1 & 4 \end{pmatrix}$ ;

d)  $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ; e)  $\begin{pmatrix} 1 & 9 & 15 \\ -5 & 5 & 9 \\ 12 & 26 & 32 \end{pmatrix}$ ;

f)  $\begin{pmatrix} a+b+c & a^2+b^2+c^2 & b^2+2ac \\ a+b+c & b^2+2ac & a^2+b^2+c^2 \\ 3. & a+b+c & a+b+c \end{pmatrix}$ .

465 a)  $\begin{pmatrix} 7 & 4 & 4 \\ 9 & 4 & 3 \\ 3 & 3 & 4 \end{pmatrix}$ ; b)  $\begin{pmatrix} 15 & 20 \\ 20 & 35 \end{pmatrix}$ ; c)  $\begin{pmatrix} 3 & -2 \\ 4 & 8 \end{pmatrix}$ ;

d)  $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ ; e)  $\begin{pmatrix} \cos n\varphi & -\sin n\varphi \\ \sin n\varphi & \cos n\varphi \end{pmatrix}$ .

466  $\begin{pmatrix} 1 & \frac{\alpha}{n} \\ -\frac{\alpha}{n} & 1 \end{pmatrix} = \sqrt{1 + \frac{\alpha^2}{n^2}} \cdot \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix}$ , where  $\operatorname{tg} \varphi = \alpha/n$ .

Thus

$$\begin{pmatrix} 1 & \frac{\alpha}{n} \\ -\frac{\alpha}{n} & 1 \end{pmatrix}^n = \left(1 + \frac{\alpha^2}{n^2}\right)^{\frac{n}{2}} \cdot \begin{pmatrix} \cos n\varphi & \sin n\varphi \\ -\sin n\varphi & \cos n\varphi \end{pmatrix}.$$

The limit of the first factor is 1. Also

$$\lim_{n \rightarrow \infty} n\varphi = \alpha \lim_{\varphi \rightarrow 0} \frac{\varphi}{\lg \varphi} = \alpha \quad . \quad \text{Thus}$$

$$\lim \begin{pmatrix} 1 & \frac{\alpha}{n} \\ -\frac{\alpha}{n} & 1 \end{pmatrix}^n = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}.$$

467 a)  $(A+B)^2 = A^2 + AB + BA + B^2 = A^2 + 2AB + B^2;$

b)  $(A+B)(A-B) = A^2 - AB + BA - B^2 = A^2 - B^2.$

c) Proof by induction.

468 a)  $\begin{pmatrix} -10 & -4 & -7 \\ 6 & 14 & 4 \\ -7 & 5 & -4 \end{pmatrix};$  b)  $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$

469 a)  $\begin{pmatrix} x & 2y \\ -y & x-2y \end{pmatrix} = (x-y)E + yA;$

b)  $\begin{pmatrix} x & y \\ 0 & x \end{pmatrix} = (x-y)E + yA;$

c)  $\begin{pmatrix} x & y & 0 \\ u & v & 0 \\ 3t-3x-u & t-3y-v & t \end{pmatrix}.$

470 a)  $\begin{pmatrix} 5 & 1 & 3 \\ 8 & 0 & 3 \\ -2 & 1 & -2 \end{pmatrix};$  b)  $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$

471 Use direct computation.

472 First note that there must exist polynomials of degree  $n^2$  that are annihilated by the matrix  $A$ . For the equation  $F(A) = a_0E + a_1A + \dots + a_mA^m = 0$  amounts to a system of  $n^2$  linear homogeneous

equations in  $m + 1$  independent variables  $a_0, a_1, \dots, a_m$ . This system has a non-trivial solution for  $m \geq n^2$ . Now let  $F(x)$  be an arbitrary polynomial annihilated by  $A$ :  $F(A) = 0$ . Let  $f(x)$  be some polynomial of lowest possible degree that has the same property. Using the Euclidean algorithm we find that  $F(x) = f(x)q(x) + r(x)$ , where the degree of  $r(x)$  is less than the degree of  $f(x)$ . But since  $r(A) = F(A) - f(A)q(A) = 0$ , the minimal property of  $f(x)$  shows that  $r$  cannot be a non-trivial polynomial:  $r(x) \equiv 0$ . Finally  $F(x) = f(x)q(x)$ .

$f$  is the minimal polynomial of  $A$ .

473 Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}; \quad B = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \cdot & \cdot & \cdot \\ b_{n1} & \dots & b_{nn} \end{pmatrix}.$$

Then the sum of the diagonal elements of the matrix  $AB$  is  $\sum_{i=1}^n \sum_{k=1}^n a_{ik}b_{ki}$ . One finds that the sum of the diagonal elements of the matrix  $BA$  is the same. Therefore, the sum of the diagonal elements of the matrix  $AB - BA$  must be zero, and the equation  $AB - BA = E$  is impossible.

Remark. This result is not valid for matrices with elements in a field of nonzero characteristic  $p$ . In fact for a field of characteristic  $p$ , the

equation  $AB - BA = E$  is satisfied by the following two  $p$ -th order matrices:

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & p-1 & 0 \end{pmatrix}.$$

474  $(E - A)(E + A + A^2 + \dots + A^{k-1}) = E - A^k = E.$

475  $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad bc = -a^2.$

476 If  $A^3 = 0$ , then  $A^2 = 0$ . For if  $A^3 = 0$ , then  $\det A = 0$ . Therefore, see problem 471,  $A^2 = (a + d)A$ ;  $0 = A^3 = (a + d)A^2 = (a + d)^2A$ . Therefore  $a + d = 0$ ,  $A^2 = 0$ .

477  $\pm E; \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad a^2 = 1 - bc.$

478 If  $A = 0$ , then  $X$  can be arbitrary. If  $\det A \neq 0$ , then  $X = 0$ . Finally, if  $\det A = 0$ , but  $A \neq 0$ , then the rows of  $A$  are proportional. Let  $\alpha : \beta$  be the ratio of corresponding elements of the first and second rows of  $A$ . Then  $X = \begin{pmatrix} -\beta x & \alpha x \\ -\beta y & \alpha y \end{pmatrix}$  where  $x, y$  are arbitrary.

479 Set  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$

1) If  $A \neq 0$  but  $a + d = 0$ ,  $ad - bc = 0$ , there is no solution.

- 2) If  $a + d \neq 0$ ,  $(a - d)^2 + 4bc = 0$ ;  $(a - d)$ ,  $b$ ,  $c$  are not identically zero, then there are two solutions:

$$X = \pm \frac{1}{2\sqrt{2(a+d)}} \begin{pmatrix} 3a+d & 2b \\ 2c & a+3d \end{pmatrix};$$

- 3) If  $a + d \neq 0$ ,  $ad - bc = 0$ , there are two solutions:

$$X = \pm \frac{1}{\sqrt{a+d}} \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

- 4) If  $ad - bc \neq 0$ ,  $(a - d)^2 + 4bc \neq 0$ , there are four solutions:

$$X = \frac{1}{\lambda} \begin{pmatrix} \frac{\lambda^2 + a - d}{2} & b \\ c & \frac{\lambda^2 - a + d}{2} \end{pmatrix},$$

where

$$\lambda = \pm \sqrt{a + d \pm 2\sqrt{ad - bc}};$$

- 5) If  $a - d = b = c = 0$ , there are infinitely many solutions:

$$X = \pm \sqrt{a}E \quad \text{or} \quad X = \begin{pmatrix} x & y \\ z & -x \end{pmatrix}, \quad \text{where } x, y, z$$

satisfy the relation  $x^2 + yz = a$ .

480 a)  $\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$ ; b)  $\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ ;

c)  $\begin{pmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$ ; d)  $\begin{pmatrix} 1 & -3 & 11 & -38 \\ 0 & 1 & -2 & 7 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ ;

$$e) \begin{pmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{pmatrix}; \quad f) \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix};$$

$$g) \begin{pmatrix} 2 & -1 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 31 & -19 & 3 & -4 \\ -23 & 14 & -2 & 3 \end{pmatrix};$$

$$h) \frac{1}{n-1} \begin{pmatrix} 2-n & 1 & 1 & \dots & 1 \\ 1 & 2-n & 1 & \dots & 1 \\ 1 & 1 & 2-n & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 2-n \end{pmatrix}$$

$$i) \frac{1}{n} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \varepsilon^{-1} & \varepsilon^{-2} & \dots & \varepsilon^{-n+1} \\ 1 & \varepsilon^{-2} & \varepsilon^{-4} & \dots & \varepsilon^{-2n+2} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \varepsilon^{-n+1} & \varepsilon^{-2n+2} & \dots & \varepsilon^{-(n-1)^2} \end{pmatrix};$$

$$j) \frac{1}{n+1} \begin{pmatrix} 1 \cdot n & 1 \cdot (n-1) & 1 \cdot (n-2) & \dots & 1 \cdot 1 \\ 1 \cdot (n-1) & 2 \cdot (n-1) & 2 \cdot (n-2) & \dots & 2 \cdot 1 \\ 1 \cdot (n-2) & 2 \cdot (n-2) & 3 \cdot (n-2) & \dots & 3 \cdot 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 \cdot 1 & 2 \cdot 1 & 3 \cdot 1 & \dots & n \cdot 1 \end{pmatrix};$$

$$k) \frac{1}{2n^3} \begin{pmatrix} 2-n^2 & 2+n^2 & 2 & \dots & 2 \\ 2 & 2-n^2 & 2+n^2 & \dots & 2 \\ \dots & \dots & \dots & \dots & \dots \\ 2+n^2 & 2 & 2 & \dots & 2-n^2 \end{pmatrix};$$

$$l) \frac{1}{d} \begin{pmatrix} b_1 c_1 + d & b_2 c_1 & \dots & b_n c_1 & -c_1 \\ b_1 c_2 & b_2 c_2 + d & \dots & b_n c_2 & -c_2 \\ \dots & \dots & \dots & \dots & \dots \\ b_1 c_n & b_2 c_n & \dots & b_n c_n + d & -c_n \\ -b_1 & -b_2 & \dots & -b_n & 1 \end{pmatrix},$$

where  $d = a - b_1 c_1 - b_2 c_2 - \dots - b_n c_n$ ;

$$\text{m) } \frac{1}{f} \begin{pmatrix} f - f_0 x^n & xf - f_1 x^n & \dots & x^{n-1}f - f_{n-1}x^n & x^n \\ -f_0 x^{n-1} & f - f_1 x^{n-1} & \dots & x^{n-2}f - f_{n-1}x^{n-1} & x^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ -f_0 x & -f_1 x & \dots & f - f_{n-1}x & x \\ -f_0 & -f_1 & \dots & -f_{n-1} & 1 \end{pmatrix},$$

where  $f_0 = a_0$ ,  $f_1 = a_0 x + a_1$ ,  $\dots$ ,  $f_{n-1} = a_0 x^{n-1} + \dots + a_{n-1}$ ,  
 $f = a_0 x^n + a_1 x^{n-1} + \dots + a_n$ ;

$$\text{n) } \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} - \frac{1}{\mu} \begin{pmatrix} \lambda_1^2 & \lambda_1 \lambda_2 & \dots & \lambda_1 \lambda_n \\ \lambda_2 \lambda_1 & \lambda_2^2 & \dots & \lambda_2 \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_n \lambda_1 & \lambda_n \lambda_2 & \dots & \lambda_n^2 \end{pmatrix},$$

$$\mu = 1 + \lambda_1 + \lambda_2 + \dots + \lambda_n;$$

$$\text{o) } \begin{pmatrix} B^{-1} + \lambda B^{-1} U V B^{-1} & -\lambda B^{-1} U \\ -\lambda V B^{-1} & \lambda \end{pmatrix}, \quad \lambda = \frac{1}{a - V B^{-1} U}.$$

481      a)  $\begin{pmatrix} 2 & -23 \\ 0 & 8 \end{pmatrix}$ ;    b)  $\begin{pmatrix} -3 & 2 & 0 \\ -4 & 5 & -2 \\ -5 & 3 & 0 \end{pmatrix}$ ;

$$\text{c) } \begin{pmatrix} 1 & -1 & -1 & 0 & \dots & 0 & 0 \\ 1 & 1 & -1 & -1 & \dots & 0 & 0 \\ 0 & 1 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 2 \end{pmatrix};$$



$$\begin{aligned} \text{d) } & \begin{pmatrix} 24 & 13 \\ -34 & -18 \end{pmatrix}; \quad \text{e) } X = E - \frac{n-1}{n^2} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \dots & 1 \end{pmatrix}; \\ \text{f) } & X = \begin{pmatrix} 1+a & b \\ -2a & 1-2b \end{pmatrix}; \quad \text{g) no solution.} \end{aligned}$$

482 Multiply both sides of the given equality on the left by  $A^{-1}$ .

$$483 \quad \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}.$$

484 If  $A^3 = E$ , then  $\det A^3 = 1$ . Therefore  $\det A = 1$ . Set  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . Then by computing  $A^{-1}$  and  $A^2$ , it is easy to conclude that either  $A = E$  or  $a + d = -1$ ,  $ad - bc = 1$ .

485 Either  $A = \pm E$  or  $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ , with  $a^2 + bc = \pm 1$ .

$$486 \quad \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = aE + bI, \quad \text{where } I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus  $I^2 = -E$ , and therefore the relation

$aE + bI \rightarrow a + bi$  is an isomorphism.

487 Set  $\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} = aE + bI + cJ + dK$ , where

$$I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \quad \text{Then}$$

$$I^2 = J^2 = K^2 = -E, \quad IJ = -JI = K, \quad JK = -KJ = I, \quad KI = -IK = J.$$

This shows that the product of two matrices of the form  $A + BI + cJ + dK$  is a matrix of the same form. This is true for sums and differences as

well as products. Thus the matrices form a ring. Moreover if we suppose  $A \neq 0$ , then

$$\det A = \begin{vmatrix} a+bi & c+di \\ -c+di & a-bi \end{vmatrix} = a^2 + b^2 + c^2 + d^2 \neq 0,$$

Thus every nonzero matrix has an inverse and the equation  $AB = 0$  is only possible if either  $A = 0$  or  $B = 0$ . The set of matrices forms not only a ring but a skew field, and is a faithful representation of the algebra of quaternions.

488 
$$(a_1E + b_1I + c_1J + d_1K)(a_2E + b_2I + c_2J + d_2K) = (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2)E + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)I + (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)J + (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)K.$$

Now take determinants and obtain the following relation

$$\begin{aligned} & (a_1^2 + b_1^2 + c_1^2 + d_1^2)(a_2^2 + b_2^2 + c_2^2 + d_2^2) = \\ & = (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2)^2 + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)^2 \\ & \quad + (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)^2 + (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)^2. \end{aligned}$$

489 The  $i$ -th and  $j$ -th rows of a matrix are permuted if it is multiplied on the left by the matrix

$$\begin{array}{c}
 i \\
 \vdots \\
 j
 \end{array}
 \begin{array}{c}
 j \\
 \vdots \\
 i
 \end{array}
 \left[ \begin{array}{ccccccc}
 1 & & & & & & \\
 & \ddots & & & & & \\
 & & 1 & & & & \\
 & & 0 & \dots & 1 & & \\
 & & \cdot & 1 & & \cdot & \\
 & & \cdot & \cdot & \cdot & \cdot & \\
 & & \cdot & & \cdot & \cdot & \\
 & & & & 1 & & \\
 & & 1 & \dots & 0 & & \\
 & & & & & 1 & \\
 & & & & & \cdot & \\
 & & & & & \cdot & \\
 & & & & & \cdot & \\
 & & & & & & 1
 \end{array} \right]$$

The  $i$ -th column of a matrix is augmented by  $\alpha$  times the  $j$ -th if it is multiplied on the left by one of the matrices

$$\left( \begin{array}{ccccccc}
 1 & & & & & & \\
 & \ddots & & & & & \\
 & & 1 & \dots & \alpha & & \\
 & & \cdot & \cdot & \cdot & \cdot & \\
 & & \cdot & & \cdot & \cdot & \\
 & & & & 1 & & \\
 & & & & & \cdot & \\
 & & & & & \cdot & \\
 & & & & & & 1
 \end{array} \right), \quad
 \left( \begin{array}{ccccccc}
 1 & & & & & & \\
 & \ddots & & & & & \\
 & & 1 & & & & \\
 & & \cdot & \cdot & \cdot & \cdot & \\
 & & \cdot & & \cdot & \cdot & \\
 & & & & \alpha & \dots & 1 \\
 & & & & & \cdot & \\
 & & & & & \cdot & \\
 & & & & & & 1
 \end{array} \right).$$

The elements of the  $i$ -th row are multiplied by the constant  $\alpha$ , if the matrix is multiplied on the left by the matrix:

$$\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \alpha & \\ & & & & 1 \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}.$$

Multiplication by the same matrices on the right accomplishes the corresponding operations on the columns of the original matrix.

- 490 The idea of the proof is to carry out elementary transformations of the type described in the preceding problem to bring the matrix  $A$  into diagonal form  $R$ . According to the preceding problem,  $R$  can be expressed in the form

$$R = U_1 U_2 \dots U_m A V_1 V_2 \dots V_k, \quad \text{where}$$

$U_1, \dots, U_m, V_1, \dots, V_k$  are the matrices that accomplish the elementary transformations. All of these are non-singular and have inverses. Therefore  $A = PRQ$ , where  $P, Q$  are non-singular.

- 491 The proof depends on the factorization in the preceding problem. By problem 489, each of the factors can be written as a product of elementary

matrices, and it is only necessary to establish the assertion for an elementary matrix of each of the three types. Indeed the first type is really superfluous since it can be written as the product of matrices of the other two types. This can be proved either by directly exhibiting a matrix of the first type as a product, or by noting that two rows can be permuted by carrying out the following operations in order: Add the first row to the second, subtract the resulting second row from the first, add the resulting first row to the second and finally multiply the first row by  $-1$ . In matrix form this equation reads:

$$E - e_{ll} - e_{kk} + e_{ik} + e_{kl} = (E - 2e_{kk})(E + e_{ik})(E - e_{kl})(E + e_{ik}).$$

The last (diagonal) matrix displayed in the solution of problem 489 can be written in the form:

$$E + (\alpha - 1) e_{11}.$$

- 492 Let  $A = P_1 R_1 Q_1$ ,  $B = P_2 R_2 Q_2$ , where  $P_1, Q_1, P_2, Q_2$  are non-singular matrices and  $R_1, R_2$  have the form  $\text{diag}\{1, 1, \dots, 1, 0, \dots, 0\}$  with  $r_1$  and  $r_2$  1's in the main diagonal respectively and zeros elsewhere. Then  $AB = P_1 R_1 Q_1 P_2 R_2 Q_2$ , and the rank of  $AB$  is the rank of  $R_1 C R_2$ ,  $C = Q_1 P_2$ . Since  $C$  is non-singular, the argument can be continued as follows. Every element in the last  $n - r_1$  rows and every element in the last  $n - r_2$  columns of  $R_1 C R_2$  is zero. Since the rank of a matrix cannot be reduced by more than one unit

when a row or a column is removed, it follows that the rank of  $R_1CR_2$  is not less than  $n - (n - r_1) - (n - r_2) = r_1 + r_2 - n$ .

493 This is a direct consequence of the fact that all the rows of a matrix of rank 1 are proportional.

494 Problem 492 shows that the rank of the matrix  $A$  is either 1 or 0. Therefore

$$A = \begin{pmatrix} \lambda_1\mu_1 & \lambda_1\mu_2 & \lambda_1\mu_3 \\ \lambda_2\mu_1 & \lambda_2\mu_2 & \lambda_2\mu_3 \\ \lambda_3\mu_1 & \lambda_3\mu_2 & \lambda_3\mu_3 \end{pmatrix}.$$

If we square the matrix we obtain the relation

$$0 = A^2 = (\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3)A,$$

from which it follows that  $\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3 = 0$ .

495 Let  $A$  be a matrix that satisfies the conditions of the problem,  $A \neq \pm E$ . Then one of the matrices  $A - E$ ,  $A + E$  has rank 1. Set

$$A + E = \begin{pmatrix} \lambda_1\mu_1 & \lambda_1\mu_2 & \lambda_1\mu_3 \\ \lambda_2\mu_1 & \lambda_2\mu_2 & \lambda_2\mu_3 \\ \lambda_3\mu_1 & \lambda_3\mu_2 & \lambda_3\mu_3 \end{pmatrix} = B.$$

Then  $A^2 = E - 2B + B^2 = E + (\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3 - 2)B$ ,

which shows that the condition  $\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3 = 2$  is necessary and sufficient for the relation  $A^2 = E$ .

The second case is handled similarly.

496 Let

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \cdot & \cdot & \cdot \\ a_{m1} & \cdots & a_{mk} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & \cdots & b_{1s} \\ \cdot & \cdot & \cdot \\ b_{m1} & \cdots & b_{ms} \end{pmatrix}, \quad C = \begin{pmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ c_m \end{pmatrix}.$$



from which it follows that  $0 \geq r_1 + r_2 - n$ .

Therefore,  $r_1 + r_2 = n$ .

- 498 The rank of the matrix  $(E + A, E - A)$  is  $n$ . Let  $B$  be a non-singular sub-matrix, the first  $r$  columns of which form a sub-matrix of  $E + A$ , and the remaining  $n - r$  columns of which form a sub-matrix of  $E - A$ . Since the relation  $(E + A)(E - A) = 0$  holds, we have:

$$(E + A)P = \begin{pmatrix} q_{11} & \cdots & q_{1r} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q_{n1} & \cdots & q_{nr} & 0 & \cdots & 0 \end{pmatrix},$$

$$(E - A)P = \begin{pmatrix} 0 & \cdots & 0 & q_{1r+1} & \cdots & q_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & q_{nr+1} & \cdots & q_{nn} \end{pmatrix}.$$

By addition, we obtain:

$$2P = \begin{pmatrix} q_{11} & \cdots & q_{1r} & q_{1r+1} & \cdots & q_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q_{n1} & \cdots & q_{nr} & q_{nr+1} & \cdots & q_{nn} \end{pmatrix}.$$

By subtraction we could have obtained:

$$2AP = \begin{pmatrix} q_{11} & \cdots & q_{1r} & -q_{1r+1} & \cdots & -q_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q_{n1} & \cdots & q_{nr} & -q_{nr+1} & \cdots & -q_{nn} \end{pmatrix} =$$

$$= 2P \begin{bmatrix} 1 & & & & & \\ & \cdot & & & & \\ & & \cdot & & & \\ & & & 1 & & \\ & & & & -1 & \\ & & & & & \cdot \\ & & & & & & -1 \end{bmatrix}.$$



The assertions above are sufficient to finish the proof.

- 499 If the elements of the matrix are integers, the value of its determinant must be an integer. If the elements of both  $A$  and  $A^{-1}$  are integers, then  $\det A$  must be a unit,  $\pm 1$ , since  $\det A \cdot \det A^{-1} = \det E = 1$ . The condition that  $\det A$  be a unit is obviously also a sufficient condition that all the elements of  $A^{-1}$  be integers, as we see for example by applying Cramer's rule to the set of equations used to determine the elements of  $A$  as unknowns.
- 500 Let  $A$  be a non-singular matrix with integral elements. The first column must contain nonzero elements. We can furthermore multiply certain rows by  $-1$  and assume that all the elements of the first column are non-negative. I claim that we can perform elementary operations on the rows and obtain a matrix, the first column of which contains  $\alpha$  in the first position and zeros elsewhere. The proof is by descent. Operate only with those rows that have nonzero entries in the first column; subtract the row with the smallest such entry from each of the other rows. If there is more than one row with a positive entry in the first column, this process will decrease the value of the largest positive entry and this remark completes the descent.

By further elementary transformations in the last  $n - 1$  rows we can produce zeros in every row below the main diagonal in the second column. Continuing in this manner, we write the original matrix as the product of a unimodular integral matrix by an upper triangular matrix.

The final condition that all the elements in the second factor be non-negative and that the diagonal element in each column be the largest, can be realized by further elementary transformations on the rows. Since the product of the matrices that bring about the elementary transformations needed in the above proof is a unimodular matrix, the assertion is established.

- 501 Set  $A = P_1 R_1 = P_2 R_2$ , where the matrices  $P_1, R_1; P_2, R_2$  satisfy the conditions of problem 500. The relation  $P_2^{-1} P_1 = R_2 R_1^{-1}$  shows that the matrix  $C = P_2^{-1} P_1$  is not only unimodular but also triangular. Set

$$R_1 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix}, \quad R_2 = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ & b_{22} & \cdots & b_{2n} \\ & & \ddots & \vdots \\ & & & b_{nn} \end{pmatrix},$$

$$C = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ & c_{22} & \cdots & c_{2n} \\ & & \ddots & \vdots \\ & & & c_{nn} \end{pmatrix}.$$

The relation  $R_2 = CR_1$  shows that

$b_{11} = c_{11}a_{11}, \dots, b_{nn} = a_{nn}c_{nn}$ . Thus every element  $c_{ii}$  is positive. But  $\det c = c_{11}c_{22} \cdots c_{nn} = \pm 1$ . Therefore,  $c_{11} = c_{22} = \cdots c_{nn} = 1$ ;  $a_{ii} = b_{ii}$ .

Next we use the relation

$b_{12} = c_{11}a_{12} + c_{12}a_{22} = a_{12} + c_{12}a_{22}$ . Solving for  $c_{12}$  we obtain  $(b_{12} - a_{12})/a_{22}$ . But from the relations  $0 \leq b_{12} < b_{22} = a_{22}$ ,  $0 \leq a_{12} < a_{22}$ , we obtain  $|c_{12}| < 1$ . Thus  $c_{12} = 0$ . Continuing in this way, from one column to another, we find that the matrix relation  $CR_1 = R_2$  can be satisfied only if all the non-diagonal elements of the matrix  $C$  are zero;  $C = E$ . Thus  $R_1 = R_2$ ,  $P_1 = P_2$ . Thus there is one and only one matrix  $R$  in each class.

The number of matrices  $R$  with given diagonal elements  $d_1, d_2, \dots, d_n$  is obviously  $d_2d_3^2 \dots d_n^{n-1}$ . Therefore the number of matrices  $R$  with given determinant  $k$  is equal to  $F_n(k) = \sum d_2d_3^2 \dots d_n^{n-1}$ , where the sum is extended over all positive integers  $d_1, d_2, \dots, d_n$  such that  $d_1d_2 \cdots d_n = k$ . If  $k = ab$ ,  $(a, b) = \text{g.c.d. of } a, b = 1$ , then every component  $d_i$  in the equation  $k = d_1d_2 \cdots d_n$  is uniquely factorable into two factors  $\alpha_i, \beta_i$  such that  $\alpha_1\alpha_2 \dots \alpha_n = a$ ,  $\beta_1\beta_2 \dots \beta_n = b$ . Therefore,

$$\begin{aligned}
 F_n(k) &= \sum_{d_1 d_2 \dots d_n = k} d_2 d_3^2 \dots d_n^{n-1} = \\
 &= \sum_{\substack{\alpha_1 \alpha_2 \dots \alpha_n = a \\ \beta_1 \beta_2 \dots \beta_n = b}} \alpha_2 \alpha_3^2 \dots \alpha_n^{n-1} \beta_2 \beta_3^2 \dots \beta_n^{n-1} = \\
 &= \sum_{\alpha_1 \alpha_2 \dots \alpha_n = a} \alpha_2 \alpha_3^2 \dots \alpha_n^{n-1} \cdot \sum_{\beta_1 \beta_2 \dots \beta_n = b} \beta_2 \beta_3^2 \dots \beta_n^{n-1} = F_n(a) \cdot F_n(b).
 \end{aligned}$$

Thus if  $k$  has the canonical decomposition

$$k = p_1^{m_1} \dots p_s^{m_s} \quad \text{it follows that}$$

$$F_n(k) = F_n(p_1^{m_1}) \dots F_n(p_s^{m_s}).$$

It remains to compute  $F_n(p^m)$ . To do

this we write the formula for  $F_n(p^m)$

as the sum of two subsums, in the first of which  $d_n = 1$ , and in the second of which  $d_n$  is divisible by  $p$ :  $d_n = p d'_n$ . Thus we obtain the formula

$$F_n(p^m) = F_{n-1}(p^m) + p^{n-1} F_n(p^{m-1})$$

From this we easily obtain, by mathematical induction, the final result.

$$F_n(p^m) = \frac{(p^{m+1}-1)(p^{m+2}-1)\dots(p^{m+n-1}-1)}{(p-1)(p^2-1)\dots(p^{n-1}-1)}.$$

502 The proof of this assertion is similar to that of problem 500, but the details are considerably less complicated. In this problem we are free to perform elementary operations on the rows and columns simultaneously. In this way we obtain a relation  $A = U_1 U_2 \dots U_r D V_1 V_2 \dots V_s$ , where

the  $U_i$  and the  $V_j$  correspond to elementary transformations. The first step is to permute the rows and columns so that the upper left-hand element has the smallest possible absolute value not equal to zero. The next step is to add or subtract the first row to the remaining rows with a view to decreasing the absolute values of the nonzero elements in the first column. At each point in this process further permutations are carried out so that the element with smallest absolute value not equal to zero is always in the upper left-hand corner. Next the first column is added to each of the other columns with the view to decreasing the absolute values of the nonzero elements in the first row, again interspersed with permutations to bring the element of smallest absolute value to the upper left corner. We see that if the original matrix was not the zero matrix, it will be possible to arrange that the upper left-hand element is not zero and the other elements in the first row and also in the first column are zero. The proof is completed by induction, that is by assuming that the theorem is true for all matrices of lower order than that of the given matrix. The final step is to notice that the product of elementary matrices is an integral unimodular matrix.

503 First note the effect of multiplying any integral matrix on the left by the matrices  $A^m$ ,  $B^m$ . If  $U$  is any matrix, the product  $A^m U$  has the same second row as  $U$ , and has a first row obtained by adding  $m$  times the second row to the first row. The product  $B^m U$  has the same first row as  $U$ , and has second row obtained by adding  $m$  times the first row to the second row.

We prove the theorem by contradiction.

Let  $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  be a matrix that cannot be

expressed as a product of powers of  $A$ ,  $B$ , and

suppose that every matrix  $U_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$

in which  $\max(a_1, c_1) < \max(a, c)$  can be expressed as a product of powers of  $A$ ,  $B$ . Then in fact  $\max(a, c) = 1$ , since otherwise we could multiply  $U$  on the left by positive and negative powers of  $A$ ,  $B$  and thus obtain a matrix expressible as the product of powers of  $A$ ,  $B$ . The proof is now essentially complete; only a small technical calculation remains. If  $a = 1$ ,  $c = 0$ , then  $U$  is already a power of  $A$ , since  $U$  is known to have determinant 1. The case  $a = c = 1$  can be reduced to the previous case if we multiply  $U$  on the left by  $B^{-1}$ . Finally, the case  $a = 0$ ,  $c = 1$  is reduced to the previous case by multiplying  $U$  on the left by  $A$ .

504 This problem follows from the preceding and two small lemmas. The first lemma asserts that  $B = CAC$ . The second lemma asserts that a matrix with determinant  $-1$  is equal to the product of another matrix with integral elements and determinant  $+1$  multiplied by the matrix  $C$ .

505 Set  $\det A = 1$ ,  $A^2 = E$ ,  $A \neq E$ . Then by problem 498:

$$A = P \cdot \text{diag}\{1, -1, -1\} \cdot P^{-1}$$

where  $P$  is some non-singular matrix. We determine the matrix  $P$  so that it is an integral matrix with smallest possible positive determinant. Since the relation  $A + E = P \cdot \text{diag}\{2, 0, 0\} \cdot P^{-1}$  holds, the matrix  $A + E$  has rank 1 and therefore

$$A + E = \begin{pmatrix} \lambda_1\mu_1 & \lambda_1\mu_2 & \lambda_1\mu_3 \\ \lambda_2\mu_1 & \lambda_2\mu_2 & \lambda_2\mu_3 \\ \lambda_3\mu_1 & \lambda_3\mu_2 & \lambda_3\mu_3 \end{pmatrix}.$$

Thus by problem 495,  $\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3 = 2$ . But since the matrix  $A + E$  is an integral matrix the numbers  $\lambda_1, \lambda_2, \lambda_3$ ,  $\mu_1, \mu_2, \mu_3$  may be taken as integers.

We now have a set of equations

$(A + E)P = P \cdot \{2, 0, 0\}$  that we can solve for the elements of the matrix  $P$ . It is not difficult to prove that  $P$  can be taken in the form

$$P = \begin{pmatrix} \lambda_1 & 0 & -\delta \\ \lambda_2 & \frac{\mu_3}{\delta} & u\mu_1 \\ \lambda_3 & -\frac{\mu_2}{\delta} & v\mu_1 \end{pmatrix},$$

where  $\delta$  is greatest common divisor of  $\mu_2, \mu_3$ ;  
and  $u, v$  are integers such that  $u\mu_2 + v\mu_3 = \delta$ .

$P$  has determinant 2.

By problem 500,  $P = QR$ , where  $Q$  is unimodular,  
and  $R$  is one of seven possible triangular matrices  
of determinant 2.

Thus  $Q^{-1}AQ$  is equal to one of the seven  
matrices  $R \cdot \text{diag}\{1, -1, -1\} \cdot R^{-1}$ . Only three  
of these matrices are really different, and two  
of these three differ only by a unimodular  
transformation. The two matrices remaining are  
those given in the problem.

506 a)  $\begin{pmatrix} 9 & 3 \\ 10 & 3 \end{pmatrix}$ ; b)  $\begin{pmatrix} 10 \\ 8 \end{pmatrix}$ ; c)  $\begin{pmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 3 & 6 & 9 \end{pmatrix}$ ; d) 13.

507 45.

508 The following identity of Euler is obtained:

$$(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2) = (aa_1 + bb_1 + cc_1)^2 + \\ + (a_1b_2 - a_2b_1)^2 + (a_1c_2 - a_2c_1)^2 + (b_1c_2 - b_2c_1)^2.$$



- 509 In the product the minor on the rows with indices  $i_1, i_2, \dots, i_m$  and columns  $k_1, k_2, \dots, k_m$  is the determinant of the product of two matrices. The first of these matrices consists of the rows of the first matrix with indices  $i$ , and the second of these matrices consists of the columns of the second factor with indices  $k$ . Therefore this minor is equal to the sum of all possible products of  $m$ -th order minors on rows  $i$  of the first matrix by corresponding minors on columns  $k$  of the second matrix.
- 510 A diagonal minor of the matrix  $A'A$  is equal to the sum of the squares of all possible minors of the same order of the matrix  $A$ . The sum runs over just the minors from the columns that involve the same indices that appear in the columns of the minor in question in the matrix  $A'A$ . Therefore it is non-negative.
- 511 If all principal  $k$ -th order minors of the matrix  $A'A$  are zero, then by problem 510, all  $k$ -th order minors of  $k$  are zero. Therefore the rank of the matrix  $A$ , and the rank of the matrix  $A'A$ , is less than  $k$ .
- 512 The sum of all the diagonal  $k$ -th order minors of either matrix  $A'A$ ,  $AA'$  is equal to the sum of the squares of all  $k$ -th order minors of the matrix  $A$ .

- 513 One multiplies the matrix  $\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix}$

by its transpose, and uses the result on the determinant of the product of two matrices.

- 514 The result is obtained by using the theorem on the determinant of a product:

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} \cdot \begin{pmatrix} a'_1 & b'_1 \\ a'_2 & b'_2 \\ \cdot & \cdot \\ a'_n & b'_n \end{pmatrix}.$$

- 515 This problem is a consequence of problem 513. Equality can occur only if the rank of the matrix

$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix}$  is less than 2, that is if the

numbers  $a_1, a_2, \dots, a_n; b_1, b_2, \dots, b_n$  are proportional.

- 516 This follows from problem 514. Equality can occur only if the numbers  $a_1, a_2, \dots, a_n; b_1, b_2, \dots, b_n$  are proportional.

- 517 Suppose the matrix  $B$  has  $m$  columns and the matrix  $C$  has  $k$  columns. By Laplace's theorem,  $\det A = \sum B_i C_i$ , where  $B_i$  runs through all possible  $m$ -th order minors of  $B$ , and  $C_i$  is the algebraic cofactor of the matrix  $B_i$ . Obviously  $C_i$  is a submatrix of  $C$ . By the Schwarz-Buniakovsky inequality of problem 515,

$(\det A)^2 \leq \sum (\det B_i)^2 \cdot \sum (\det C_i)^2$ . But  
 $\sum (\det B_i)^2 = \det B'B$ .

518 Set

$$B = \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{pmatrix}, \quad C = \begin{pmatrix} c_{11} & \cdots & c_{1k} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nk} \end{pmatrix}; \quad A = (B, C).$$

The inequality is trivial in the case  $m + k > n$ , and follows from problem 517 if  $m + k = n$ . The remaining case,  $m + k < n$  is treated as follows. First suppose that, for arbitrary  $j, s$ , the equality

$$\sum_{i=1}^n b_{ij} c_{is} = 0$$

holds. Then

$$A'A = \begin{pmatrix} B'B & 0 \\ 0 & C'C \end{pmatrix} \text{ and therefore}$$

$$\det A'A = \det B'B \cdot \det C'C.$$

Without loss of generality we can assume that  $\text{rank } A = m + k$ , because if this is not true the inequality is trivial.

We must append a matrix  $D$  to  $A$  so that the matrix  $(A, D)$  is square and so that  $D$  has rank  $n - m - k$ . We also demand that the inner product of any column of  $D$  by any column of  $A$  be zero.

This can be accomplished as follows. First obtain the square matrix  $\tilde{\epsilon}' = (A, \tilde{D}')$  choosing  $\tilde{D}'$  so that  $\tilde{\epsilon}'$  is non-singular. Next, replace each element of  $\tilde{D}'$  by its algebraic cofactor in the matrix  $\tilde{\epsilon}'$ . It should be clear that the rank of the matrix  $D$  so obtained is equal to the number  $n - m - k$  of its columns, since  $D$  is a submatrix of the matrix whose elements are the algebraic cofactors of the matrix  $\tilde{\epsilon}'$ , and the latter differs from the matrix  $\tilde{\epsilon}'^{-1}$  by the constant factor  $\det \tilde{\epsilon}$ . Thus we have shown how to obtain the matrix  $D$  needed in the preceding paragraph.

Set  $P = (A, D)$ ;  $Q = (C, D)$ . By problem 517 we have  $\det P'P \leq \det B'B \cdot \det Q'Q$ . But  $\det P'P = \det A'A \cdot \det D'D$ ;  $\det Q'Q = \det C'C \cdot \det D'D$ . Since the factor  $\det D'D$  is positive, we can factor it out and obtain the result

$$\det A'A \leq \det B'B \cdot \det C'C.$$

- 519 The result follows if we apply problem 518 to the matrix  $A'$ .
- 520 The determinant of the matrix  $A^*A$  is the sum of the squares of the moduli of all  $m$ -th order minors of  $A$ , where  $m$  is the number of columns of the matrix  $A$ .
- 521 The solution follows the method used to solve problems 517, 518. For a square matrix one uses Laplace's expansion and the Schwarz-Buniakovsky

inequality. For a rectangular matrix one appends further columns to form a square matrix in such a way that the inner product of each of the appended rows by any row of  $A$  is 0. (Note: The inner product of two rows is given by the formula  $a_1 \bar{b}_1 + a_2 \bar{b}_2 + \dots + a_m \bar{b}_m$ .)

- 522 In problem 521, let  $B$  be one of the columns of  $A$ . Apply problem 521 several times and obtain:

$$\det(A^*A) = |\det A|^2 \leq \sum_{i=1}^n |a_{i1}|^2 \cdot \sum_{i=1}^n |a_{i2}|^2 \dots \sum_{i=1}^n |a_{in}|^2 \leq n^n M^{2n}$$

thus

$$|\det A| \leq n^{\frac{n}{2}} M^n.$$

- 523 Border the given matrix by a column of  $n + 1$  elements on the left, all equal to  $M/2$ , and a row of  $n$  zeros above. Let  $\Delta_1$  be the determinant of the new matrix,  $\Delta$  be the determinant of the original matrix. Then  $\Delta = 2\Delta_1/M$ . Now subtract the first column of the enlarged matrix from each of the other columns. In the matrix so obtained no element has absolute value exceeding  $M/2$ . The assertion now follows by applying the result of problem 522.

524 Let  $\varepsilon = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$ . The matrix

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & \varepsilon & \dots & \varepsilon^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \varepsilon^{n-1} & \dots & \varepsilon^{(n-1)^2} \end{bmatrix}$$

has determinant  $n^{n/2}$ .

525 For  $n = 2^m$ , an  $n$ -th order matrix can be constructed as a Kronecker product of the matrix  $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

with itself  $m$  times. For  $m = 2$  the matrix obtained is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Since the inner product of every two rows of such a matrix is 0 one finds:

$$A'A = \begin{pmatrix} n & 0 & \dots & 0 \\ 0 & n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & n \end{pmatrix}; \det(A'A) = n^n; |\det A| = n^{n/2}.$$

If we multiply every element of the matrix  $A$  by the constant  $M$ , we have the obvious equality:

$$|\det (MA)| = M^n n^{n/2}.$$

- 526 To see that the absolute value of the determinant can be increased (or at least not decreased) by replacing the element in question by  $+1$  or  $-1$ , it is sufficient to examine the expansion of the determinant by minors of a row in which the element in question stands. In this expansion, if the cofactor of the element in question has the same sign as the value of the determinant, the element may be replaced by  $+1$ ; in the opposite case by  $-1$ .

By multiplying appropriate rows by  $-1$  if necessary, we can assume that all the elements of the first column are  $+1$ . Except for the first element we can also assume that all the elements in the first row are  $-1$ . Now add the first column to each of the remaining columns, and expand the determinant of the matrix by minors of the first row. Since every element beyond the first column in the second, third, fourth, ...,  $n$ -th row is divisible by 2 (being equal to zero or 2), the determinant of the matrix will be divisible by  $2^{n-1}$ .

The proof above shows in fact that the determinant of an  $n \times n$  matrix of 1's and  $-1$ 's must be divisible by  $2^{n-1}$ . Possible values of this determinant are therefore of the form  $2^{n-1}N$ , where  $N$  is the determinant of an  $n-1 \times n-1$  matrix of 0's and 1's. The complete set of values that  $N$  may have is not known except in the following cases.

<u>n-1</u>	<u>N</u>
1	0 or 1
2	0 or 1
3	0, 1, or 2
4	0, 1, 2, or 3
5	0, 1, 2, 3, 4, or 5

527 4 for  $n = 3$ ; 48 for  $n = 5$ .

528 The result is trivial if  $A$  is a singular matrix. Therefore let  $A$  be nonsingular and  $A'$  be its transpose; let  $\Delta$  be the determinant; let  $A_1$  be the obverse of  $A$ . Then  $A_1 = \Delta CA'^{-1}C$ , where  $C = \text{diag}\{-1, 1, -1, \dots\}$ . This is clear from the formula for the elements of the inverse of the matrix. Therefore

$$\det A_1 = \Delta^{n-1}; (A_1)_1 = \Delta^{n-1}C(A'_1)^{-1}C = \Delta^{n-1} \cdot \Delta^{-1}A = \Delta^{n-2}A.$$

529 We use the following notation. Let the minor of the obverse matrix  $A_1$  involve the rows  $i_1 < i_2 < \dots < i_m$  and columns  $k_1 < k_2 < \dots < k_m$ . Let the numbers of the remaining rows be  $i_{m+1} < i_{m+2} < \dots < i_n$ ; and the numbers of the remaining columns be  $k_{m+1} < k_{m+2} < \dots < k_n$ . The product of the determinant  $\Delta$  of the matrix  $A$  by the determinant of the minor being considered satisfies the following relations:



$$\begin{aligned}
& \Delta \cdot \begin{vmatrix} \Delta_{i_1 k_1} & \cdots & \Delta_{i_1 k_m} \\ \vdots & \ddots & \vdots \\ \Delta_{i_m k_1} & \cdots & \Delta_{i_m k_m} \end{vmatrix} \\
&= (-1)^{i_1 + \cdots + i_m + k_1 + \cdots + k_m} \Delta \cdot \begin{vmatrix} A_{i_1 k_1} & \cdots & A_{i_m k_1} \\ \vdots & \ddots & \vdots \\ A_{i_1 k_m} & \cdots & A_{i_m k_m} \end{vmatrix} \\
&= \begin{vmatrix} A_{i_1 k_1} & \cdots & A_{i_m k_1} & A_{i_{m+1} k_1} & \cdots & A_{i_n k_1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{i_1 k_m} & \cdots & A_{i_m k_m} & A_{i_{m+1} k_m} & \cdots & A_{i_n k_m} \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{vmatrix} \cdot \begin{vmatrix} a_{i_1 k_1} & \cdots & a_{i_1 k_m} & \cdots & a_{i_1 k_n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i_m k_1} & \cdots & a_{i_m k_m} & \cdots & a_{i_m k_n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i_n k_1} & \cdots & a_{i_n k_m} & \cdots & a_{i_n k_n} \end{vmatrix} \\
&= \begin{vmatrix} \Delta & & & & & \\ & \ddots & & & & \\ & & \ddots & & & \\ & & & \Delta & & \\ a_{i_{m+1} k_1} & \cdots & a_{i_{m+1} k_{m+1}} & \cdots & a_{i_{m+1} k_n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i_n k_1} & \cdots & a_{i_n k_{m+1}} & \cdots & a_{i_n k_n} \end{vmatrix} \\
&= \Delta^m \cdot \begin{vmatrix} a_{i_{m+1} k_{m+1}} & \cdots & a_{i_{m+1} k_n} \\ \vdots & \ddots & \vdots \\ a_{i_n k_{m+1}} & \cdots & a_{i_n k_n} \end{vmatrix}
\end{aligned}$$

This relation proves the assertion.

530, These results follow from the theorem on the  
 531 determinant of the product of two rectangular  
 matrices.

532 The correct ordering is the lexicographic one. This means that the sequence  $i_1 < i_2 < \dots < i_m$  precedes the sequence  $j_1 < j_2 < \dots < j_m$  if the first non-zero difference  $i_1 - j_1, i_2 - j_2, \dots$  is positive. Then every minor of a triangular matrix has determinant zero if its column index precedes its row index in the above ordering.

533 Exercises 531, 491 indicate that it is sufficient to establish the theorem for a triangular matrix. By Exercise 532, if  $A$  is a triangular matrix, then

$$\det A_{(m)} = \prod_{i_1 < i_2 < \dots < i_m} a_{i_1 i_1} a_{i_2 i_2} \dots a_{i_m i_m} = (\det A)^{\binom{n-1}{m-1}}$$

534 Properties a, b follow immediately from the definition. We establish property c by noting the following formulas, where the notation for elements of the respective matrices is the usual one involving upper and lower case Roman letters. If

$$C = (A' \cdot A'') \times (B' \cdot B''), \quad A' \times B' = G, \quad A'' \times B'' = H,$$

then

$$\begin{aligned} c_{i_1 k_1, i_2 k_2} &= \sum_{i=1}^n a'_{i i_1} a''_{i i_2} \cdot \sum_{k=1}^m b'_{k i_1} b''_{k i_2} = \sum_{i, h} a'_{i i_1} b'_{h i_1} a''_{i i_2} b''_{h i_2} \\ &= \sum_{i, h} g_{i i_1, h i_1} h_{i i_2, h i_2} \end{aligned}$$

Thus  $C = G \cdot H$ , which establishes the assertion.

535 A verbal proof depends on the fact that the rows and columns of the Kronecker product  $A \times B$  can

be permuted to give a matrix which obviously has the determinant asserted.

Another proof depends on the relation

$$A \times B = (A \times E_m) \cdot (E_n \times B).$$

But the determinant of the first factor  $A \times E_m$  is the same as the determinant of the matrix

$$\text{diag}\{A, A, \dots, A\},$$

as is seen by permuting the rows of columns. This remark is sufficient to prove the assertion.

- 536 In the matrix  $C_{ik}$ , the element in the  $\alpha$ -th row and  $\beta$ -th column is

$$\begin{aligned} c_{(i-1)m+\alpha, (k-1)m+\beta} &= \sum_{s=1}^{mn} a_{(i-1)m+\alpha, s} b_{s, (k-1)m+\beta} \\ &= \sum_{j=1}^n \sum_{v=1}^m a_{(i-1)m+\alpha, (j-1)m+v} b_{(j-1)m+v, (k-1)m+\beta}. \end{aligned}$$

Now examine the above formula. The inner sum in the last term is the element of the  $\alpha$ -th row and  $\beta$ -th column of the matrix  $A_{ij}B_{jk}$ . This proves that

$$C_{ik} = \sum_{j=1}^n A_{ij}B_{jk}.$$

- 537 The theorem is trivial for  $n = 1$ . We prove the theorem by induction, suppose that it has been proved for a matrix of order  $n - 1$  and use the induction hypothesis to establish the theorem for matrix of order  $n$ . Here "order" refers to the number of boxes.

First suppose that the sub-matrix  $A_{11}$  is non-singular:

$$C = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}.$$

Let  $D$  be given by the formula

$$D = \begin{pmatrix} E, -A_{11}^{-1}A_{12}, \dots, -A_{11}^{-1}A_{1n} \\ E & & & \\ & \cdot & & \\ & & \cdot & \\ & & & E \end{pmatrix},$$

and form the product  $C' = CD$ :

$$C' = \begin{pmatrix} A_{11} & 0 & \dots & 0 \\ A_{21} & A'_{22} & \dots & A'_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A'_{n2} & \dots & A'_{nn} \end{pmatrix},$$

where  $A'_{ik} = A_{ik} - A_{i1}A_{11}^{-1}A_{1k}$ .

The explicitly written sub-matrices in the displays for  $C$ ,  $D$ ,  $C'$  are commutative. This being the case it is easy to check that the formula for the determinant of a product of two matrices remains valid when one considers only formal determinants of "matrices," the elements of which are the displayed sub-matrices.

The formal determinant of the matrix  $D$  is  $E^n$ ; the actual determinant of  $D$  is one.

Thus

$$\det C = \det C' = (\det A_{11}) \cdot \det \begin{bmatrix} A'_{22} & \dots & A'_{2n} \\ \vdots & \ddots & \vdots \\ A'_{n2} & \dots & A'_{nn} \end{bmatrix}.$$

Let  $B$  be the formal determinant. Then we have  $B = A_{11} \cdot B'$ , where  $B'$  is the formal determinant of the matrix

$$\begin{pmatrix} A'_{22} & \dots & A'_{2n} \\ \vdots & \ddots & \vdots \\ A'_{n2} & \dots & A'_{nn} \end{pmatrix}.$$

The induction hypothesis however shows that

$$\det B' = \det \begin{bmatrix} A'_{22} & \dots & A'_{2n} \\ \vdots & \ddots & \vdots \\ A'_{n2} & \dots & A'_{nn} \end{bmatrix},$$

and therefore  $\det B = \det A_{11} \cdot \det B' = \det C$ , which establishes the assertion.

If the matrix  $A_{11}$  has zero determinant, the proof can be modified by considering the matrix

$$C(\lambda) = \begin{pmatrix} A_{11} + \lambda E_m & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}$$

and by denoting its formal determinant by  $B(\lambda)$ .

Note that, for almost all values of  $\lambda$ , the relation  $\det C(\lambda) = \det B(\lambda)$  holds, and that  $\det (A_{11} + \lambda E_m) = \lambda^m + \dots \neq 0$ , also holds. These relations could not be valid unless the relation  $\det C = \det B$  were valid. Thus the assertion is completely established.

The theorem in question remains valid when the elements of the sub-matrices belong to an arbitrary ring. The theorem is due to Ingraham. See also a recent paper by Afriat.

[Ingraham, M.H. A Note on Determinants, Bull. Amer. Math. Soc. 43 (1937), 579-580; and Afriat, S.N. Composite Matrices, Q. J. Math., Oxford Ser(2) 5, (1954), 81-98.]

CHAPTER V - SOLUTIONS  
POLYNOMIALS AND RATIONAL FUNCTIONS  
OF A SINGLE INDETERMINATE

- 538      a)  $2x^6 - 7x^5 + 6x^4 - 3x^3 - x^2 - 2x + 1$ ;  
b)  $x^6 + x^5 - 3x^4 - 4x^3 + x^2 + 3x + 1$ .
- 539      a) Quotient  $2x^2 + 3x + 11$ , remainder  $25x - 5$ .  
b) Quotient  $\frac{3x-7}{9}$ , remainder  $\frac{-26x-2}{9}$ .
- 540       $p = -q^2 - 1$ ,  $m = q$ .
- 541      1)  $q = p - 1$ ,  $m = 0$ ; 2)  $q = 1$ ,  $m = \pm \sqrt{2 - p}$ .
- 542       $(-1)^n \frac{(x-1)(x-2)\dots(x-n)}{1 \cdot 2 \cdot 3 \dots n}$ .
- 543      a)  $(x-1)(x^3 - x^2 + 3x - 3) + 5$ ;  
b)  $(x+3)(2x^4 - 6x^3 + 13x^2 - 39x + 109) - 327$ ;  
c)  $(x+1+i)[4x^2 - (3+4i)x + (-1+7i)] + 8 - 6i$ ;  
d)  $(x-1+2i)[x^2 - 2ix - 5 - 2i] - 9 + 8i$ .
- 544      a) 136;    b)  $-1 - 44i$ .
- 545      a)  $(x+1)^4 - 2(x+1)^3 - 3(x+1)^2 + 4(x+1) + 1$ ;  
b)  $(x-1)^5 + 5(x-1)^4 + 10(x-1)^3 + 10(x-1)^2 + 5(x-1) + 1$ ;  
c)  $(x-2)^4 - 18(x-2) + 38$ ;  
d)  $(x+i)^4 - 2i(x+i)^3 - (1+i)(x+i)^2 - 5(x+i) + 7 + 5i$ ;  
e)  $(x+1-2i)^4 - (x+1-2i)^3 + 2(x+1-2i) + 1$ .

546      a)  $\frac{1}{(x-2)^2} + \frac{6}{(x-2)^3} + \frac{11}{(x-2)^4} + \frac{7}{(x-2)^5};$   
           b)  $\frac{1}{x+1} - \frac{4}{(x+1)^2} + \frac{4}{(x+1)^3} + \frac{2}{(x+1)^5}.$

547      a)  $x^4 + 11x^3 + 45x^2 + 81x + 55;$   
           b)  $x^4 - 4x^3 + 6x^2 + 2x + 8.$

548      a)  $f(2) = 18, \quad f'(2) = 48, \quad f''(2) = 124, \quad f'''(2) = 216,$   
            $f^{IV}(2) = 240, \quad f^V(2) = 120;$   
           b)  $f(1+2i) = -12 - 2i, \quad f'(1+2i) = -16 + 8i, \quad f''(1+2i) =$   
            $= -8 + 30i, \quad f'''(1+2i) = 24 + 30i, \quad f^{IV}(1+2i) = 24.$

549      a) 3;    b) 4.

550       $a = -5.$

551       $A = 3, \quad B = -4.$

552       $A = n, \quad B = -(n + 1).$

555      A necessary and sufficient condition that

$(x - 1)^{k+1}$  divide  $f(x)$  is that

$f(1) = a_0 + a_1 + \cdots + a_n = 0$ , and that  $f'(x)$  be divisible by  $(x - 1)^k$ . We now observe that the polynomial  $f_1(x) = nf(x) - xf'(x)$  will be divisible by  $(x - 1)^k$  but not by a higher power of  $x - 1$ . One treats  $f_1(x)$  as a polynomial of formal degree  $n$ , and repeats the same argument  $k$  times.

556       $a$  is a  $k + 3$ -fold root, where  $k$  is the multiplicity of  $a$  as a root of  $f'''(x)$ .



This shows that the numbers  $a_1 x^{p_1}, a_2 x^{p_2}, \dots, a_k x^{p_k}$  are proportional to the algebraic cofactors of the elements of the last row of the Vandermonde matrix

It is easy to check that

$$\frac{\det \Delta}{\det \Delta_1} = \prod_{s \neq 1} (p_1 - p_s) = \varphi'(p_1).$$

This shows that the numbers  $a_i x^{p_i}$  are inversely proportional to  $\varphi'(p_i)$ ; thus

$$a_1 x^{p_1} \varphi'(p_1) = a_2 x^{p_2} \varphi'(p_2) = \dots = a_k x^{p_k} \varphi'(p_k).$$

The converse is established by inverting the order of argument at each step.

- 563 If  $f(x)$  is divisible by  $f'(x)$ , then the quotient must be a first degree polynomial with highest coefficient  $1/n$ , where  $n$  is the degree of  $f(x)$ . Thus  $nf(x) = (x - x_0) f'(x)$ . Continued differentiation gives

$$(n-1) f'(x) = (x - x_0) f''(x), \dots,$$

thus

$$f(x) = \frac{(x - x_0)^n}{n!} f^{(n)}(x) = a_0 (x - x_0)^n.$$

The converse is obvious.

- 564 A multiple root of the polynomial

$f(x) = 1 + \frac{x}{1} + \dots + \frac{x^n}{n!}$  will be a root of its derivative also:

$$f'(x) = 1 + \frac{x}{1} + \dots + \frac{x^{n-1}}{(n-1)!} = f(x) - \frac{x^n}{n!}.$$

Thus if  $f(x_0) = f'(x_0)$ , then  $x_0 = 0$ . But 0 is not a root of  $f(x)$ .

- 565 If  $f(x) = (x - x_0)^k f_1(x)$ , where  $f_1(x)$  is a rational function that is not zero for  $x = x_0$ , then successive differentiation gives

$$f(x_0) = f'(x_0) = \dots = f^{(k-1)}(x_0) = 0, \quad f^{(k)}(x_0) \neq 0.$$

Conversely if  $f(x_0) = f'(x_0) = \dots = f^{(k-1)}(x_0) = 0$ ,  $f^{(k)}(x_0) \neq 0$ , then  $f(x) = (x - x_0)^k f_1(x)$ ,  $f_1(x_0) \neq 0$ . For if the relation

$$f(x) = (x - x_0)^m q(x), \quad q(x_0) \neq 0$$

were valid for some  $m \neq k$ , then the number of successive derivatives that vanished at  $x = x_0$  would be greater or less than  $k$ .

- 566 The function

$$g(x) = \frac{\psi(x)}{w(x)} = f(x) - f(x_0) - \frac{f'(x_0)}{1}(x - x_0) - \dots \\ \dots - \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

satisfies the conditions

$$g(x_0) = g'(x_0) = \dots = g^{(n)}(x_0) = 0.$$

Therefore,  $\psi(x) = (x - x_0)^{n+1} F(x)$ , where  $F(x)$  is a polynomial, which proves the assertion.

- 567 If  $f_1(x)f_2(x_0) - f_2(x)f_1(x_0)$  is not identically equal to 0, then we can suppose that  $f_1(x_0) \neq 0$ .

Consider the rational function  $\frac{f_2(x)}{f_1(x)} - \frac{f_2(x_0)}{f_1(x_0)}$ .

It has  $x_0$  for a root and is not identically zero.

The multiplicity is one greater than the multiplicity of  $x_0$  as a root of the derivative. The latter is

equal to  $\frac{f_1(x)f_2'(x) - f_2(x)f_1'(x)}{[f_1(x)]^2}$ . The assertion

follows from this fact.

- 568 Let  $x_0$  be a  $k$ -fold root of  $[f'(x)]^2 - f(x)f''(x)$ . Then  $f(x_0) \neq 0$ , for  $x_0$  cannot be a root of both  $f(x)$  and  $f'(x)$ . By assumption  $x_0$  is a  $k+1$  fold root of the polynomial  $f(x)f'(x_0) - f(x_0)f'(x)$ , and the degree of this polynomial does not exceed  $n$ . Therefore,  $k+1 \leq n$ ,  $k \leq n-1$ .
- 569 The polynomial  $f(x)f'(x_0) - f(x_0)f'(x)$  must have  $x_0$  as an  $n$ -fold root, that is must be of the form  $A(x-x_0)^n$ , where  $A$  is a constant. Set  $z = x - x_0$ , and expand in powers of  $z$ :
- $$(a_0 + a_1z + a_2z^2 + \dots + a_nz^n)a_1 - (a_1 + 2a_2z + 3a_3z^2 + \dots + na_nz^{n-1})a_0 = Az^n,$$
- where
- $$a_0 = f(x_0) \neq 0.$$
- Thus  $a_2 = \frac{a_1^2}{2a_0}$ ,  $a_3 = \frac{a_1^3}{a_0^2 3!}$ ,  $\dots$ ,  $a_n = \frac{a_1^n}{a_0^{n-1} n!}$ .
- Setting  $\frac{a_1}{a_0} = \alpha$ , we obtain
- $$f(x) = a_0 \left[ 1 + \frac{\alpha(x-x_0)}{1} + \frac{\alpha^2(x-x_0)^2}{1 \cdot 2} + \dots + \frac{\alpha^n(x-x_0)^n}{n!} \right].$$
- 570 For example,  $\delta = 1/21$  ( $1/20$  does not suffice).
- 571 For example,  $\delta = 1/25$  ( $1/24$  does not suffice).
- 572 For example,  $M = 6$ .

573 For example,

$$a) x = \rho i, \quad 0 < \rho < \sqrt[3]{\frac{4}{3}};$$

$$b) x = \rho, \quad 0 < \rho < \sqrt[3]{\frac{4}{3}}.$$

574 For example, a)  $x = 1 - \rho$ ,  $0 < \rho < \frac{1}{8}$ ;

$$b) x = 1 + \rho \left( \cos \frac{(2k-1)\pi}{4} + i \sin \frac{(2k-1)\pi}{4} \right), \quad \rho < \sqrt[4]{8};$$

$$c) x = 1 + \rho i, \quad \rho < \frac{1}{\sqrt{3}}.$$

575 Expand the polynomial  $f(z)$  in powers of  $h = z - i$  to obtain

$$f(z) = (2-i) \left[ 1 + (1-i)h^3 - \frac{4+2i}{5}h^4 + \frac{1+3i}{5}h^5 \right].$$

If we substitute  $h = a(1-i)$ , we obtain

$$f(z) = (2-i) \left[ 1 - 4a^3 + 4a^3 \left( \frac{4+2i}{5}a - \frac{4+2i}{5}a^2 \right) \right],$$

so that

$$|f(z)| \leq \sqrt{5} \left( |1 - 4a^3| + 4a^3 \frac{1}{4} \sqrt{\frac{4}{5}} \right) < \sqrt{5}$$

for

$$0 < a < \frac{1}{2}.$$

576 We write the polynomial in the form

$$f(z) = f(z_0) \{ 1 + r(\cos \varphi + i \sin \varphi)(z - z_0)^k [1 + (z - z_0)\psi(z)] \},$$

$$\text{set } z - z_0 = \rho(\cos \theta + i \sin \theta), \text{ choose } \theta = \frac{2m\pi - \varphi}{k}$$

and choose  $\rho$  small enough so that  $|(z - z_0)\psi(z)| < 1$ .

Then

$$|f(z)| = |f(z_0)| |1 + r\rho^k + r\rho^k(z - z_0)\psi(z)| > |f(z_0)|.$$

577 The proof is similar to that for polynomials and uses Taylor's formula for rational functions, problem 566. Write this formula so that it includes the first term after  $f(x_0)$  with non-zero coefficient.

578 Let  $M$  be the greatest lower bound for  $|f(z)|$  in the given region of the  $z$ -plane. It can be proved that there is a point  $z_0$  in this plane, in every neighborhood of which  $|f(z)|$  has  $M$  as greatest lower bound, by continually subdividing the region. The highest possible power of  $z - z_0$  is to be cancelled from numerator and denominator. Call the numerator and denominator in the result  $\varphi(z)$ ,  $\psi(z)$  respectively:  
 $f(z) = \varphi(z)/\psi(z)$ . Then  $\psi(z_0) \neq 0$ , since otherwise the least upper bound of  $|f(z)|$  could not exist in sufficiently small neighborhoods of  $z_0$ .  
 Therefore  $f(z)$  is continuous at  $z = z_0$ , from which it follows that  $|f(z_0)| = M$ , as it was to be proved.

579 A non-constant rational function does not necessarily have infinite modulus for large values of the independent variable.

580 The hypothesis gives the relations

$$f(a) \neq 0, \quad f'(a) = \dots = f^{(k-1)}(a) = 0, \quad f^{(k)}(a) \neq 0$$

and by Taylor's formula, we have

$$f(z) = f(a) + \frac{f^{(k)}(a)}{k!} (z-a)^k [1 + \varphi(z)], \quad \varphi(a) = 0.$$

Now set

$$\frac{1}{f(a)} \cdot \frac{f^{(k)}(a)}{k!} = r(\cos \varphi + i \sin \varphi), \quad z - a = \rho(\cos \theta + i \sin \theta).$$

Suppose  $\rho$  to be chosen small enough so that

$$|\varphi(z)| < 1, \quad r\rho^k < 1. \quad \text{Then}$$

$$|f(z)| = |f(a)| \cdot |1 + r\rho^k [\cos(\varphi + k\theta) + i \sin(\varphi + k\theta)] + r\rho^k \lambda|,$$

where  $|\lambda| < 1$ . The following relations are clearly valid.

$$\text{A. } |f(z)| < |f(a)| \quad \text{for } \theta = \frac{(2m-1)\pi - \varphi}{k}$$

$$\text{B. } |f(z)| > |f(a)| \quad \text{for } \theta = \frac{2m\pi - \varphi}{k}$$

Therefore the value of  $|f(z)| - |f(a)|$  changes sign  $2k$  times as  $\theta$  increases from

$$\{(\pi - \varphi)/k\} \text{ to } \{(\pi - \varphi)/k\} + 2\pi. \quad \text{But}$$

$$|f(z)| - |f(a)| \text{ is a continuous function of } \theta$$

and must therefore reduce to 0

$2k$  times, as was to be shown.

581 By the method used in the preceding exercise we can show that if  $\rho$  is sufficiently small, then as  $\theta$  increases by  $2\pi$ , and  $z = \rho(\cos \theta + i \sin \theta)$ , the quantities  $\operatorname{Re}\{f(z) - f(a)\}$ ,  $\operatorname{Im}\{f(z) - f(a)\}$  change sign  $2k$  times. We set  $f^{(k)}(a)/k! = r(\cos \varphi + i \sin \varphi)$  and have, by Taylor's formula,  $f(z) - f(a) = r\rho^k [\cos(\varphi + k\theta) + i \sin(\varphi + k\theta)] [1 + \varphi(z)]$ ,  $\varphi(a) = 0$ . Now set  $\varphi(z) = \varphi_1(z) + i\varphi_2(z)$  and choose  $\rho$  so that  $|\varphi(z)| < 1$ . Then

$$\begin{aligned} \operatorname{Re}(f(z)) - \operatorname{Re}(f(a)) = & r\rho^k [\cos(\varphi + k\theta)(1 + \varphi_1(z)) \\ & - \sin(\varphi + k\theta)\varphi_2(z)]; \end{aligned}$$

$$\operatorname{Im}(f(z)) - \operatorname{Im}(f(a)) = r\rho^k [\sin(\varphi + k\theta)(1 + \varphi_1(z)) + \cos(\varphi + k\theta)\varphi_2(z)].$$

We now choose  $\theta$  so that

$$\varphi + k\theta = m\pi, \quad m = 0, 1, 2, \dots, 2k,$$

and obtain

$$\operatorname{Re}(f(z)) - \operatorname{Re}(f(a)) = r\rho^k (-1)^m (1 + \varepsilon_m),$$

where  $\varepsilon_m$  is the corresponding value of

$$\varphi_1(z), \quad |\varepsilon_m| < 1.$$

This shows that  $\operatorname{Re}\{f(z) - f(a)\}$  changes sign  $2m$  times when  $z$  traces the circumference  $|z - a| = \rho$ . In a similar manner we can find that  $\operatorname{Im}\{f(z) - f(a)\}$  changes sign  $2m$  times, by putting  $\varphi + k\theta = \frac{\pi}{2} + m\pi$ ,  $m = 0, 1, \dots, 2k$ .

582

- a)  $(x-1)(x-2)(x-3);$   
 b)  $(x-1-i)(x-1+i)(x+1-i)(x+1+i);$   
 c)  $\left(x+1 - \sqrt{\frac{\sqrt{2}+1}{2}} - i\sqrt{\frac{\sqrt{2}-1}{2}}\right) \times \left(x+1 - \sqrt{\frac{\sqrt{2}+1}{2}} + i\sqrt{\frac{\sqrt{2}-1}{2}}\right) \times \left(x+1 + \sqrt{\frac{\sqrt{2}+1}{2}} + i\sqrt{\frac{\sqrt{2}-1}{2}}\right) \times \left(x+1 + \sqrt{\frac{\sqrt{2}+1}{2}} - i\sqrt{\frac{\sqrt{2}-1}{2}}\right);$   
 d)  $(x - \sqrt{3} - \sqrt{2})(x - \sqrt{3} + \sqrt{2})(x + \sqrt{3} - \sqrt{2}) \times (x + \sqrt{3} + \sqrt{2}).$



583

$$\begin{aligned} \text{a) } & 2^{n-1} \prod_{k=1}^n \left( x - \cos \frac{(2k-1)\pi}{2n} \right); \\ \text{b) } & 2 \prod_{k=1}^n \left( x + \frac{\sin \left( \theta + \frac{(2k-1)\pi}{2n} \right)}{\sin \frac{(2k-1)\pi}{2n}} \right); \quad \text{c) } \prod_{k=1}^m \left( x - \operatorname{tg}^2 \frac{(2k-1)\pi}{4m} \right). \end{aligned}$$

584

$$\begin{aligned} \text{a) } & (x^2 + 2x + 2)(x^2 - 2x + 2); \\ \text{b) } & (x^2 + 3)(x^2 + 3x + 3)(x^2 - 3x + 3); \\ \text{c) } & \left( x^2 + 2x + 1 + \sqrt{2} - 2(x+1)\sqrt{\frac{\sqrt{2}+1}{2}} \right) \\ & \quad \times \left( x^2 + 2x + 1 + \sqrt{2} + 2(x+1)\sqrt{\frac{\sqrt{2}+1}{2}} \right); \\ \text{d) } & \prod_{k=0}^{n-1} \left( x^2 - 2\sqrt[2n]{2} x \cos \frac{(8k+1)\pi}{4n} + \sqrt[n]{2} \right); \\ \text{e) } & (x^2 - x\sqrt{a+2} + 1)(x^2 + x\sqrt{a+2} + 1); \\ \text{f) } & \prod_{k=0}^{n-1} \left( x^2 - 2x \cos \frac{(3k+1)2\pi}{3n} + 1 \right). \end{aligned}$$

585

$$\begin{aligned} \text{a) } & (x-1)^2(x-2)(x-3)(x-1-l) \\ & = x^5 - (8+i)x^4 + (24+7i)x^3 - (34+17i)x^2 + (23+17i)x \\ & \quad - (6+6i); \\ \text{b) } & (x+1)^3(x-3)(x-4) = x^5 - 4x^4 - 6x^3 + 16x^2 + 29x + 12; \\ \text{c) } & (x-l)^2(x+1+l) = x^3 + (1-l)x^2 + (1-2l)x - 1-l. \end{aligned}$$

586

$$\prod_{k=1}^n X_k(x).$$

- 587 a)  $(x-1)^2(x-2)(x-3)(x^2-2x+2) = x^6 - 9x^5 + 33x^4 - 65x^3 + 74x^2 - 46x + 12$ ;  
 b)  $(x^2-4x+13)^3 = x^6 - 12x^5 + 87x^4 - 376x^3 + 1131x^2 - 2028x + 2197$ ;  
 c)  $(x^2+1)^2(x^2+2x+2) = x^6 + 2x^5 + 4x^4 + 4x^3 + 5x^2 + 2x + 2$ .
- 588 a)  $(x-1)^2(x+2)$ ; b)  $(x+1)^2(x^2+1)$ ; c)  $(x-1)^3$ .
- 589  $x^d - 1$ , where  $d = (m, n)$ .
- 590  $x^d + a^d$ , if  $m/d, n/d$  are odd; 1, if either is even;  
 $d = (m, n)$ .
- 591 a)  $(x-1)^2(x+1)$ ; b)  $(x-1)^3(x+1)$ ;  
 c)  $x^d - 1$ ,  $d = (m, n) = \text{g.c.d. of } m, n$ .
- 592 Set  $\lambda_0 = u(x_0)/v(x_0)$ , and factor  $f(x)$  into linear factors:  $f(x) = (x-\lambda_0)(x-\lambda_1)\dots(x-\lambda_{k-1})$ .

Then, since  $\lambda_j \neq \lambda_0$ , we find

$$f\left(\frac{u(x)}{v(x)}\right) = \frac{1}{[v(x)]^k} (u(x) - \lambda_0 v(x)) \dots (u(x) - \lambda_{k-1} v(x)).$$

Now note that since  $u(x_0) - \lambda_j v(x_0) = v(x_0)(\lambda_0 - \lambda_j) \neq 0$ , the hypotheses of the theorem show that  $u(x) - \lambda_0 v(x)$  has  $x_0$  as a  $k$ -fold root,  $k > 1$ . Therefore,

$u'(x) - \lambda_0 v'(x)$  has  $x_0$  as a  $k-1$ -fold root. Thus,

$$f\left(\frac{u'(x)}{v'(x)}\right) = \frac{1}{[v'(x)]^k} (u'(x) - \lambda_0 v'(x)) \dots (u'(x) - \lambda_{k-1} v'(x)).$$

It is obvious that for  $j \neq 0$ , the factors

$u'(x) - \lambda_j v'(x)$ , are all nonzero. Therefore,

$f\left(\frac{u'(x)}{v'(x)}\right)$  has  $x_0$  as a  $k-1$ -fold root, as was to be shown.

593 If  $\omega$  is a root of  $x^2 + x + 1$ , then  $\omega^3 = 1$ .

Therefore  $\omega^{3m} + \omega^{3n+1} + \omega^{3p+2} = 1 + \omega + \omega^2 = 0$ .

594 If  $\lambda$  is a root of  $x^2 - x + 1$ , then  $\lambda^3 = -1$ .

Therefore

$$\begin{aligned}\lambda^{3m} - \lambda^{3n+1} + \lambda^{3p+2} &= (-1)^m - (-1)^n \lambda + (-1)^p \lambda^2 \\ &= (-1)^m - (-1)^p + \lambda [(-1)^p - (-1)^n].\end{aligned}$$

The last relation shows that the result can be zero only if  $(-1)^m = (-1)^p = (-1)^n$ , that is if  $m, n, p$  are all even or all odd.

595  $x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 - x + 1)$ . The two factors are relatively prime, and  $x^2 + x + 1$  always divides  $x^{3m} + x^{3n+1} + x^{3p+2}$  by problem 593. It remains to determine when  $x^2 - x + 1$  is also a divisor. The method used to solve the preceding problem gives

$$(-1)^m + (-1)^n \lambda + (-1)^p \lambda^2 = (-1)^m - (-1)^p + \lambda [(-1)^n + (-1)^p].$$

Thus we must have  $(-1)^m = (-1)^p = -(-1)^n$ . Thus the numbers  $m, p, n + 1$  must be all even or all odd.

596 If  $m$  is not divisible by 3.

597 All roots of the polynomial  $x^{k-1} + x^{k-2} + \dots + 1$  are  $k$ -th roots of 1. Therefore the relation

$$\xi^{ka_1} + \xi^{ka_2+1} + \dots + \xi^{ka_k+k-1} = 1 + \xi + \dots + \xi^{k-1} = 0$$

holds. Thus divisibility does occur since all the roots of the first polynomial are simple.

- 598 When a root  $\omega$  of the polynomial  $x^2 + x + 1$  is substituted into the polynomial  $f(x) = (1 + x)^m - x^m - 1$ , the result obtained is  $(1 + \omega)^m - \omega^m - 1$ . But  $1 + \omega = -\omega^2 = \lambda$  is a primitive 6-th root of 1. Moreover,  $\omega = \lambda^2$ ; thus  $f(\omega) = \lambda^m - \lambda^{2m} - 1$ .

For

$$\begin{aligned} m = 6n & & f(\omega) &= -1 \neq 0; \\ m = 6n + 1 & & f(\omega) &= \lambda - \lambda^2 - 1 = 0; \\ m = 6n + 2 & & f(\omega) &= \lambda^2 + \lambda - 1 \neq 0; \\ m = 6n + 3 & & f(\omega) &= -3 \neq 0; \\ m = 6n + 4 & & f(\omega) &= -\lambda + \lambda^2 - 1 \neq 0; \\ m = 6n + 5 & & f(\omega) &= -\lambda^2 + \lambda - 1 = 0. \end{aligned}$$

$f(x)$  is divisible by  $x^2 + x + 1$  if  $m = 6n + 1$  or  $m = 6n + 5$ .

- 599 For  $m = 6n + 2$  and  $m = 6n + 4$ .

- 600  $f(\omega) = m(1 + \omega)^{m-1} - m\omega^{m-1} = m[\lambda^{m-1} - \lambda^{2(m-1)}]$ ,  $f'(\omega) = 0$  only for  $m = 6n + 1$ .

- 601 For  $m = 6n + 4$ .

- 602 No, since the first and second derivatives are not simultaneously equal to 0.

- 603 For  $x = k$ ,  $1 \leq k \leq n$ , we have

$$f(k) = 1 - \frac{k}{1} + \frac{k(k-1)}{1 \cdot 2} - \dots + (-1)^k \frac{k(k-1) \dots 1}{1 \cdot 2 \dots k} = (1-1)^k = 0.$$

Thus, the polynomial is divisible by

$(x-1)(x-2) \dots (x-n)$ . Comparing the highest coefficients, we see that

$$f(x) = \frac{(-1)^n}{n!} (x-1)(x-2) \dots (x-n).$$

- 604 For the case  $(m, n) = 1$ .
- 605 If  $f(x^n)$  is divisible by  $x - 1$ , then  $f(1) = 0$ .  
Therefore  $f(x)$  is divisible by  $x - 1$  and  $f(x^n)$  is divisible by  $x^n - 1$ .
- 606 If  $F(x) = f(x^n)$  is divisible by  $(x - a)^k$ , then  
 $F'(x) = f'(x^n) n x^{n-1}$  is divisible by  $(x - a)^{k-1}$ .  
 Thus  $f'(x^n)$  is divisible by  $(x - a)^{k-1}$ .  
 By the same argument,  $f''(x^n)$  is divisible by  $(x - a)^{k-2}$ , ..., and  $f^{(k-1)}(x^n)$  is divisible by  $x - a$ .  
 Thus we find that  $f(a^n) = f'(a^n) = \dots = f^{(k-1)}(a^n) = 0$ .  
 Therefore  $f(x)$  is divisible by  $(x - a^n)^k$ ;  $f(x^n)$  is divisible by  $(x^n - a^n)^k$ .
- 607 If  $F(x) = f_1(x^3) + x f_2(x^3)$  is divisible by  $x^2 + x + 1$ , then  $F(\omega) = f_1(1) + \omega f_2(1) = 0$   
 where  $\omega$  is a root of  $x^2 + x + 1$ . Thus  
 $F(\omega^2) = f_1(1) + \omega^2 f_2(1) = 0$ , and finally  
 $f_1(1) = f_2(1) = 0$ .
- 608 The polynomial  $f(x)$  has no real root of odd multiplicity since otherwise it would change sign. Therefore  $f(x) = [f_1(x)]^2 f_2(x)$ , where  $f_2(x)$  is a polynomial with no real roots. The complex roots of the polynomial  $f_2$  can be classified into two sets, those in one set being complex conjugates of those in the other. Using the linear factors corresponding to the roots in the two sets, we find that  $f_2(x)$  is a product of two complex conjugate polynomials:

$$\psi_1(x) + i\psi_2(x) \quad , \quad \psi_1(x) - i\psi_2(x).$$

Therefore

$$f_2(x) = \psi_1^2(x) + \psi_2^2(x) \quad , \quad f(x) = (f_1\psi_1)^2 + (f_1\psi_2)^2.$$

609      a)  $-x_1, -x_2, \dots, -x_n$ ; b)  $\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}$ ;  
c)  $x_1 - a, x_2 - a, \dots, x_n - a$ ; d)  $bx_1, bx_2, \dots, bx_n$ .

610      One of the roots must be  $-p/2$ . This gives the relation  $84 = 4pq - p^3$ .

611       $x_1 = 1/6, \quad x_2 = 1/2, \quad x_3 = -1/3$ .

612       $a^3 - 4ab + 8c = 0$ .

613      The relation among the roots must be valid for arbitrary  $\alpha$ . Setting  $\alpha = -a/4$ , we obtain the transformed equation

$$y^4 + a'y^3 + b'y^2 + c'y + d' = 0, \quad a' = 0, \quad a'^3 - 4a'b' + 8c' = 0, \quad \text{thus } c' = 0.$$

614       $a^2d = c^2$ .

615      Division by  $x^2$  gives  $x^2 + \frac{d}{x^2} + a\left(x + \frac{c}{ax}\right) + b = 0$ .

Now set  $x + \frac{c}{ax} = z$ , and obtain

$$x^2 + \frac{d}{x^2} = x^2 + \frac{c}{a^2x^2} = z^2 - 2\frac{c}{a},$$

therefore the original equation can be written as a quadratic in  $z$ :  $z^2 + az + b - 2\frac{c}{a} = 0$ . It

is easy to find  $x$  after solving for  $z$   
(generalized reciprocal equation).

$$616 \quad \begin{array}{l} \text{a) } x = 1 \pm \sqrt{3}, 1 \pm i\sqrt{2}; \quad \text{b) } x = 1 \pm 2i, -2 \pm i; \\ \text{c) } x = \frac{-1 \pm \sqrt{5}}{2}, \frac{-1 \pm i\sqrt{11}}{2}; \quad \text{d) } x = 1 \pm \sqrt{3}, \frac{-3 \pm \sqrt{17}}{2}. \end{array}$$

$$617 \quad \lambda = \pm 6.$$

$$618 \quad 1) b = c = 0, \quad a \text{ arbitrary};$$

$$2) a = -1, b = -1, c = 1.$$

$$619 \quad 1) a = b = c = 0; \quad 2) a = 1, b = -2, c = 0;$$

$$3) a = 1, b = -1, c = -1; \quad 4) b = \lambda, a = -\frac{1}{\lambda}, \\ c = \frac{2 - \lambda^2}{\lambda}, \text{ where } \lambda^3 - 2\lambda + 2 = 0.$$

$$620 \quad \lambda = -3.$$

$$621 \quad q^3 + pq + q = 0.$$

$$622 \quad a_1^2 - 2a_2.$$

$$623 \quad x_i = -\frac{a_1}{n} + \frac{2i-n-1}{2}h, \quad i = 1, 2, \dots, n, \quad \text{where}$$

$$h = \frac{1}{n} \sqrt{\frac{12(n-1)a_1^2 - 24na_2}{n^2 - 1}}.$$

624 If the roots lie in arithmetic progression, then  
the formula of the preceding exercise gives:

a)  $-1/2, 1/2, 3/2$  which do satisfy the given  
equation;

b)  $-5/2, -3/2, -1/2, 1/2$  which do not satisfy  
the given equation.

- 625 Let  $y = Ax + B$  be the equation of the given line. Then the roots of the equation  $x^4 + ax^3 + bx^2 + cx + d = Ax + B$  must lie in arithmetic progression. By problem 623 we can write them

$$x_i = -\frac{a}{4} + \frac{2i-5}{2}h, \quad i = 1, 2, 3, 4,$$

where

$$h = \frac{1}{2} \sqrt{\frac{9a^2 - 24b}{15}} = \frac{1}{2} \sqrt{\frac{3a^2 - 8b}{5}}.$$

Moreover

$$\begin{aligned} A - c &= x_1x_4(x_2 + x_3) + x_2x_3(x_1 + x_4) = \\ &= -\left(\frac{a^2}{16} - \frac{9}{4}h^2\right)\frac{a}{2} - \left(\frac{a^2}{16} - \frac{1}{4}h^2\right)\frac{a}{2} = \frac{a^3 - 4ab}{8}, \\ d - B &= x_1x_2x_3x_4 = \frac{1}{1600}(36b - 11a^2)(4b + a^2). \end{aligned}$$

Thus

$$A = \frac{a^3 - 4ab + 8c}{8}, \quad B = d - \frac{1}{1600}(36b - 11a^2)(4b + a^2).$$

The points of intersection will be real and distinct if  $3a^2 - 8b > 0$ , that is if the second derivative  $2(6x^2 + 3ax + b)$  changes sign as  $x$  traverses the real axis.

626  $x^4 - ax^2 + 1 = 0$ , where  $a = \frac{\alpha^4 + 1}{\alpha^2}$ .

627  $(x^2 - x + 1)^3 - a(x^2 - x)^2 = 0$ ,  $a = \frac{(a^2 - \alpha + 1)^3}{(a^2 - \alpha)^2}$ .

628  $f'(x_i) = (x_i - x_1) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_n);$   
 $f''(x_i) = 2[(x_i - x_2) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_n)$   
 $+ (x_i - x_1)(x_i - x_3) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_n)$   
 $+ \dots + (x_i - x_1)(x_i - x_2) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots$   
 $\dots (x_i - x_{n-1})] = 2f'(x_i) \sum_{\substack{s=1 \\ (s \neq i)}}^n \frac{1}{x_i - x_s} \quad (\text{if } x_s \neq x_i).$



629 Follows immediately from problem 628.

630 Set  $x_i = x_1 + (i - 1)h$ . Then

$$f'(x_i) = (-1)^{n-i} (i-1)! (n-i)! h^{n-1}.$$

631 a)  $x+1$ ; b)  $x^2+1$ ; c)  $x^3+1$ ; d)  $x^2-2x+2$ ;  
e)  $x^3-x+1$ ; f)  $x+3$ ; g)  $x^2+x+1$ ; h)  $x^2-2x\sqrt{2}-1$ ;  
i)  $x+2$ ; j) 1; k)  $2x^2+x-1$ ; l)  $x^2+x+1$ .

632 a)  $(-x-1)f_1(x) + (x+2)f_2(x) = x^2-2$ ;  
b)  $-f_1(x) + (x+1)f_2(x) = x^3+1$ ;  
c)  $(3-x)f_1(x) + (x^2-4x+4)f_2(x) = x^2+5$ ;  
d)  $(1-x^2)f_1(x) + (x^3+2x^2-x-1)f_2(x) = x^3+2$ ;  
e)  $(-x^2+x+1)f_1(x) + (x^3+2x^2-5x-4)f_2(x) = 3x+2$ ;  
f)  $-\frac{x-1}{3}f_1(x) + \frac{2x^2-2x-3}{3}f_2(x) = x-1$ .

633 a)  $M_2(x) = x$ ,  $M_1(x) = -3x^2 - x + 1$ ;  
b)  $M_2(x) = -x-1$ ,  $M_1(x) = x^3 + x^2 - 3x - 2$ ;  
c)  $M_2(x) = \frac{-x^2+3}{2}$ ,  $M_1(x) = \frac{x^4-2x^2-2}{2}$ ;  
d)  $M_2(x) = -\frac{2x^2+3x}{6}$ ,  $M_1(x) = \frac{2x^3+5x^2-6}{6}$ ;  
e)  $M_2(x) = 3x^2+x-1$ ,  $M_1(x) = -3x^3+2x^2+x-2$ ;  
f)  $M_2(x) = -x^3-3x^2-4x-2$ ,  
 $M_1(x) = x^4+6x^3+14x^2+15x+7$ .

634 a)  $M_2(x) = \frac{-16x^2+37x+26}{3}$ ,  $M_1(x) = \frac{16x^3-53x^2-37x-23}{3}$ ;  
b)  $M_2(x) = 4-3x$ ,  $M_1(x) = 1+2x+3x^2$ ;  
c)  $M_2(x) = 35-84x+70x^2-20x^3$ ,  
 $M_1(x) = 1+4x+10x^2+20x^3$ .

635 a)  $M_1(x) = 9x^2-26x-21$ ,  
 $M_2(x) = -9x^3+44x^2-39x-7$ ;  
b)  $M_1(x) = 3x^3+3x^2-7x+2$ ,  
 $M_2(x) = -3x^3-6x^2+x+2$ .

- 636 a)  $4x^4 - 27x^3 + 66x^2 - 65x + 24$ ;  
 b)  $-5x^7 + 13x^6 + 27x^5 - 130x^4 + 75x^3 + 266x^2 - 440x + 197$ .

637 
$$N(x) = 1 + \frac{n}{1}x + \frac{n(n+1)}{1 \cdot 2}x^2 + \dots$$

$$\dots + \frac{n(n+1) \dots (n+m-2)}{1 \cdot 2 \dots (m-1)}x^{m-1};$$

$$M(x) = 1 + \frac{m}{1}(1-x) + \frac{m(m+1)}{1 \cdot 2}(1-x)^2 + \dots$$

$$\dots + \frac{m(m+1) \dots (m+n-2)}{1 \cdot 2 \dots (n-1)}(1-x)^{n-1}$$

$$= \frac{(m+1)(m+2) \dots (m+n-1)}{(n-1)!} - \frac{m}{1} \frac{(m+2) \dots (m+n-1)}{(n-2)!}x$$

$$+ \frac{m(m+1)}{1 \cdot 2} \frac{(m+3) \dots (m+n-1)}{(n-3)!}x^2 - \dots$$

$$\dots + (-1)^{n-1} \frac{m(m+1) \dots (m+n-2)}{(n-1)!}x^{n-1}.$$

638 1.

- 639 a)  $(x+1)^4(x-2)^2$  b)  $(x+1)^4(x-4)$ ;  
 c)  $(x-1)^3(x+3)^2(x-3)$ ; d)  $(x-2)(x^2-2x+2)^2$ ;  
 e)  $(x^3-x^2-x-2)^2$ ; f)  $(x^2+1)^2(x-1)^3$ ;  
 g)  $(x^4+x^3+2x^2+x+1)^2$ .

- 640 a)  $f(x) = x + 1 + \frac{1}{24}x(x-1)(x-2)(x-3)$ ;  
 b)  $f(x) = -x^4 + 4x^3 - x^2 - 7x + 5$ ;  
 c)  $f(x) = 1 + \frac{2}{5}(x-1) - \frac{1}{105}(x-1)(4x-9) +$   
 $+ \frac{1}{945}(x-1)(4x-9)(x-4),$   
 $f(2) = 1 \frac{389}{945} = 1,4116 \dots \quad (\sqrt{2} = 1,4142 \dots);$   
 d)  $f(x) = x^3 - 9x^2 + 21x - 8.$

$$\begin{aligned}
 641 \quad & \text{a) } y = -\frac{1}{3}(x-2)(x-3)(x-4) \\
 & + \frac{1}{2}(x-1)(x-3)(x-4) - 2(x-1)(x-2)(x-4) \\
 & + \frac{1}{2}(x-1)(x-2)(x-3) = -\frac{4}{3}x^3 + 10x^2 - \frac{65}{3}x + 15; \\
 & \text{b) } y = \frac{1}{2}[5 - (1-i)x - x^2 - (1+i)x^3].
 \end{aligned}$$

$$642 \quad f(x) = \frac{n+1}{2} - \frac{1}{2} \sum_{k=1}^{n-1} \left(1 - i \operatorname{ctg} \frac{k\pi}{n}\right) x^k.$$

Solution.

$$\begin{aligned}
 f(x) &= \sum_{s=0}^{n-1} \frac{(s+1)(x^n-1)}{(x-\varepsilon_s)n\varepsilon_s^{n-1}} \\
 &= \frac{1}{n} \sum_{s=0}^{n-1} \frac{(s+1)(1-x^n)}{(1-x\varepsilon_s^{-1})} = \frac{1}{n} \sum_{s=0}^{n-1} \sum_{k=0}^{n-1} (s+1)x^k\varepsilon_1^{-ks} \\
 &= \frac{1}{n} \sum_{k=0}^{n-1} x^k \sum_{s=0}^{n-1} (s+1)\varepsilon_1^{-ks} = \frac{1}{n} \sum_{s=0}^{n-1} (s+1) \\
 &+ \frac{1}{n} \sum_{k=1}^{n-1} x^k \sum_{s=0}^{n-1} (s+1)\varepsilon_k^{-s} = \frac{n+1}{2} - \sum_{k=1}^{n-1} \frac{x^k}{1-\varepsilon_k^{-1}} \\
 &= \frac{n+1}{2} - \frac{1}{2} \sum_{k=1}^{n-1} \left(1 - i \operatorname{ctg} \frac{k\pi}{n}\right) x^k.
 \end{aligned}$$

$$643 \quad f(x) = \sum_{k=1}^n \frac{y_k(x^n-1)}{(x-\varepsilon_k)n\varepsilon_k^{n-1}} = \frac{1}{n} \sum_{k=1}^n \frac{y_k(1-x^n)}{1-x\varepsilon_k^{-1}}, \quad f(0) = \frac{1}{n} \sum_{k=1}^n y_k.$$

644 Set  $\varphi(x) = (x-x_1)(x-x_2)\dots(x-x_n)$ . Let  $f(x)$  be an arbitrary polynomial of degree  $n-1$  or less; let it assume the values  $y_1, y_2, \dots, y_n$  for  $x = x_1, x_2, \dots, x_n$ . Then

$$f(x_0) = \frac{y_1 + y_2 + \dots + y_n}{n} = \sum_{k=1}^n \frac{y_k \varphi(x_0)}{\varphi'(x_k)(x_0 - x_k)}.$$

Since  $y_1, y_2, \dots, y_n$  are arbitrary, then

$$\frac{\varphi(x_0)}{\varphi'(x_i)(x_0 - x_i)} = \frac{1}{n}.$$

Consider the polynomial

$$F(x) = n[\varphi(x_0) - \varphi(x)] - (x_0 - x)\varphi'(x).$$

Its degree is  $< n$ , and it takes the value 0 for

$x = x_1, x_2, \dots, x_n$ . Thus,  $F(x) = 0$ . Expand

$\varphi(x)$  in powers of  $(x - x_0)$ :

$$\varphi(x) = \sum_{k=0}^n c_k (x - x_0)^k.$$

We find  $\sum_{k=1}^n (n-k)c_k (x - x_0)^k = 0$ . Thus

$$c_1 = c_2 = \dots = c_{n-1} = 0;$$

$$\varphi(x) = (x - x_0)^n + c_0, \quad x_i = x_0 + \sqrt[n]{-c_0}.$$

$$645 \quad x^s = \sum_{i=1}^n \frac{x_i^s \varphi(x)}{(x - x_i) \varphi'(x_i)}.$$

Comparing coefficients (of  $x^{n-1}$ ) we obtain

$$\sum_{i=1}^n \frac{x_i^s}{\varphi'(x_i)} = 0.$$

$$646 \quad x^{n-1} = \sum_{i=1}^n \frac{x_i^{n-1} \varphi(x)}{(x - x_i) \varphi'(x_i)}.$$

Comparing coefficients (of  $x^{n-1}$ ) we obtain

$$\sum_{i=1}^n \frac{x_i^{n-1}}{\varphi'(x_i)} = 1.$$

$$647 \quad a_i = \frac{1}{\Delta} \sum_{k=1}^n y_k \Delta_{ki}, \quad \text{where}$$

$$\Delta = \det \begin{vmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ 1 & x_2 & \dots & x_2^{n-1} \\ \dots & \dots & \dots & \dots \\ 1 & x_n & \dots & x_n^{n-1} \end{vmatrix},$$

$\Delta_{ki}$  is the algebraic cofactor of the element in the  $k$ -th row and  $i + 1$ -th column in the matrix above.

$$f(x) = \sum_{i=0}^{n-1} a_i x^i = \frac{1}{\Delta} \sum_{k=1}^n y_k \sum_{i=0}^{n-1} \Delta_{ki} x^i = \sum_{k=1}^n y_k \frac{\Delta_k}{\Delta},$$

where  $\Delta_k$  is the determinant of the matrix obtained from the matrix above by replacing the  $k$ -th row by  $1, x, \dots, x^{n-1}$ .

Expanding the determinants,  $\Delta_k, \Delta$  as Vandermonde determinants gives

$$\frac{\Delta_k}{\Delta} = \frac{(x-x_1) \dots (x-x_{k-1})(x-x_{k+1}) \dots (x-x_n)}{(x_k-x_1) \dots (x_k-x_{k-1})(x_k-x_{k+1}) \dots (x_k-x_n)} = \frac{\varphi(x)}{(x-x_k) \varphi'(x)},$$

where  $\varphi(x) = (x-x_1)(x-x_2) \dots (x-x_n)$ .

Thus  $f(x) = \sum \frac{y_k \varphi(x)}{(x-x_k) \varphi'(x_k)}$ , which was to be proved.

$$648 \quad f(x) = 1 + \frac{x}{1!} + \frac{x(x-1)}{2!} + \dots + \frac{x(x-1) \dots (x-n+1)}{n!}.$$

$$649 \quad f(x) = 1 + \frac{(a-1)x}{1} + \frac{(a-1)^2 x(x-1)}{1 \cdot 2} + \dots \\ \dots + \frac{(a-1)^n x(x-1) \dots (x-n+1)}{n!}.$$

$$650 \quad f(x) = 1 - \frac{2x}{1} + \frac{2x(2x-2)}{1 \cdot 2} + \dots \\ \dots + \frac{2x(2x-2) \dots (2x-4n+2)}{(2n)!}.$$

$$651 \quad f(x) = 1 - \frac{x-1}{2!} + \frac{(x-1)(x-2)}{3!} - \dots \\ \dots + (-1)^n \frac{(x-1)(x-2) \dots (x-n+1)}{n!} = \frac{n! - (1-x)(2-x) \dots (n-x)}{n! x}.$$

652  $f(x) = \frac{\varphi(a) - \varphi(x)}{\varphi(a)(x-a)}$ , where  $\varphi(x) = (x-x_1)(x-x_2)\dots(x-x_n)$ .

653 We write  $f(x)$  in the form

$$f(x) = A_0 + A_1 \frac{x-m}{1} + A_2 \frac{(x-m)(x-m-1)}{1 \cdot 2} + \dots \\ \dots + A_n \frac{(x-m)(x-m-1)\dots(x-m-n+1)}{n!},$$

where  $m, m+1, \dots, m+n$  are the integral values of  $x$  for which the polynomial  $f(x)$  takes on integral values.

Replacing  $x$  by  $m, m+1, \dots, m+n$ , we obtain the following set of equations for determining  $A_0, A_1, \dots, A_n$ :

$$A_0 = f(m), \\ A_k = f(m+k) - A_0 - \frac{k}{1} A_1 - \frac{k(k-1)}{1 \cdot 2} A_2 - \dots - k A_{k-1}, \\ k = 1, 2, \dots, n,$$

From these equations it is clear that all coefficients  $A_k$  have integral values. When  $x$  has any integral value, each term of  $f(x)$  is the product of a binomial coefficient by an integer  $A_k$ , and therefore is itself an integer. Therefore,  $f(x)$  takes on integral values for integral  $x$ , as was to be proved.

654 The polynomial  $F(x) = f(x^2)$  of degree  $2n$  takes on integral values for the  $2n+1$  values  $x = -n, -(n-1), \dots, -1, 0, 1, \dots, n$ , and by the preceding problem takes on integral values for all integral values of  $x$ .

655

- a)  $\frac{1}{12(x-1)} - \frac{4}{3(x+2)} + \frac{9}{4(x+3)}$ ;  
 b)  $\frac{1}{6(x-1)} + \frac{1}{2(x-2)} - \frac{1}{2(x-3)} + \frac{1}{6(x-4)}$ ;  
 c)  $\frac{2}{x-1} + \frac{-2+i}{2(x-i)} + \frac{-2-i}{2(x+i)}$ ;  
 d)  $\frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{i}{4(x-i)} + \frac{i}{4(x+i)}$ ;  
 e)  $\frac{1}{3} \left( \frac{1}{x-1} + \frac{\varepsilon}{x-\varepsilon} + \frac{\varepsilon^2}{x-\varepsilon^2} \right), \quad \varepsilon = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ ;  
 f)  $-\frac{1}{16} \left( \frac{1+i}{x-1-i} + \frac{1-i}{x-1+i} + \frac{-1+i}{x+1-i} + \frac{-1-i}{x+1+i} \right)$ ;  
 g)  $\frac{1}{n} \sum_{k=0}^{n-1} \frac{\varepsilon_k}{x-\varepsilon_k}, \quad \varepsilon_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$ ;  
 h)  $-\frac{1}{n} \sum_{k=1}^n \frac{\eta_k}{x-\eta_k}, \quad \eta_k = \cos \frac{(2k-1)\pi}{n} + i \sin \frac{(2k-1)\pi}{n}$ ;  
 i)  $\sum_{k=0}^n \frac{C_n^k (-1)^{n-k}}{x-k}$ ; j)  $\sum_{k=-n}^n \frac{(-1)^{n-k} C_{2n}^{n+k}}{x-k}$ ;  
 k)  $\frac{1}{n} \sum_{k=1}^n \frac{(-1)^{k-1} \sin \frac{2k-1}{2n} \pi}{x - \cos \frac{2k-1}{2n} \pi}.$

656

- a)  $\frac{1}{3(x-1)} - \frac{x+2}{3(x^2+x+1)}$ ;  
 b)  $\frac{1}{8(x-2)} - \frac{1}{8(x+2)} + \frac{1}{2(x^2+4)}$ ;  
 c)  $\frac{1}{8} \frac{x+2}{x^2+2x+2} - \frac{1}{8} \frac{x-2}{x^2-2x+2}$ ;  
 d)  $\frac{1}{18} \left( \frac{1}{x^2+3x+3} + \frac{1}{x^2-3x+3} - \frac{2}{x^2+3} \right)$ ;  
 e)  $\frac{1}{2n+1} \left[ \frac{1}{x-1} + 2 \sum_{k=1}^n \frac{x \cos \frac{2k(m+1)\pi}{2n+1} - \cos \frac{2km\pi}{2n+1}}{x^2 - 2x \cos \frac{2k\pi}{2n+1} + 1} \right];$

$$f) \frac{(-1)^m}{2n+1} \left[ \frac{1}{x+1} + 2 \sum_{k=1}^n \frac{x \cos \frac{2k(m+1)\pi}{2n+1} + \cos \frac{2km\pi}{2n+1}}{x^2 + 2x \cos \frac{2k\pi}{2n+1} + 1} \right];$$

$$g) \frac{1}{2n} \left[ \frac{1}{x-1} - \frac{1}{x+1} + 2 \sum_{k=1}^{n-1} \frac{x \cos \frac{k\pi}{n} - 1}{x^2 - 2x \cos \frac{k\pi}{n} + 1} \right];$$

$$h) \frac{1}{n} \sum_{k=1}^n \frac{\cos \frac{(2k-1)m\pi}{n} - x \cos \frac{(2k-1)(2m+1)\pi}{2n}}{x^2 - 2x \cos \frac{(2k-1)\pi}{2n} + 1};$$

$$i) \frac{1}{(n!)^2 x} + 2 \sum_{k=1}^n \frac{(-1)^k x}{(n+k)!(n-k)!(x^2 + k^2)}.$$

657

$$a) \frac{1}{4(x-1)^2} - \frac{1}{4(x+1)^2};$$

$$b) \frac{1}{4(x+1)} - \frac{1}{4(x-1)} + \frac{1}{4(x-1)^2} + \frac{1}{4(x+1)^2};$$

$$c) \frac{3}{(x-1)^3} - \frac{4}{(x-1)^2} + \frac{1}{x-1} - \frac{1}{(x+1)^2} - \frac{2}{x+1} + \frac{1}{x-2};$$

$$d) \frac{1}{n^2} \left[ \sum_{k=0}^{n-1} \frac{\varepsilon_k^2}{(x - \varepsilon_k)^2} - (n-1) \sum_{k=0}^{n-1} \frac{\varepsilon_k}{(x - \varepsilon_k)} \right],$$

$$\varepsilon_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n};$$

$$e) \frac{1}{x^m} + \frac{\frac{n}{1}}{x^{m-1}} + \frac{\frac{n(n+1)}{1 \cdot 2}}{x^{m-2}} + \dots + \frac{\frac{n(n+1) \dots (n+m-2)}{1 \cdot 2 \dots (m-1)}}{x} \\ + \frac{\frac{m}{1}}{(1-x)^n} + \frac{\frac{m}{1}}{(1-x)^{n-2}} + \frac{\frac{m(m+1)}{1 \cdot 2}}{(1-x)^{n-2}} + \dots \\ \dots + \frac{\frac{m(m+1) \dots (m+n-2)}{1 \cdot 2 \dots (n-1)}}{1-x};$$

$$f) \frac{1}{(-4a^2)^n} \sum_{k=0}^{n-1} (2a)^{n-k} \frac{n(n+1) \dots (n+k-1)}{k!} \\ \times \left[ \frac{1}{(a-x)^{n-k}} + \frac{1}{(a+x)^{n-k}} \right];$$



$$g) \frac{1}{(4a^2)^n} \sum_{k=0}^{n-1} (2a)^{n-k} \frac{n(n+1) \dots (n+k-1)}{k!} \times \left[ \frac{1}{(a-ix)^{n-k}} + \frac{1}{(a+ix)^{n-k}} \right];$$

$$h) \sum_{k=1}^n \frac{g(x_k)}{[f'(x_k)]^2 (x-x_k)^2} + \sum_{k=1}^n \frac{g'(x_k) f'(x_k) - g(x_k) f''(x_k)}{[f'(x_k)]^3 (x-x_k)}.$$

658

$$\begin{aligned} a) & -\frac{1}{4(x+1)} + \frac{x-1}{4(x^2+1)} + \frac{x+1}{2(x^2+1)^2}; \\ b) & -\frac{1}{x} + \frac{7}{x+1} + \frac{3}{(x+1)^2} - \frac{6x+2}{x^2+x+1} - \frac{3x+2}{(x^2+x+1)^2}; \\ c) & \frac{1}{16(x-1)^2} - \frac{3}{16(x-1)} + \frac{1}{16(x+1)^2} + \frac{3}{16(x+1)} + \\ & \quad + \frac{1}{4(x^2+1)} + \frac{1}{4(x^2+1)^2}; \\ d) & \frac{1}{4n^2} \left[ \frac{1}{(x-1)^2} + \frac{1}{(x+1)^2} - \frac{2n-1}{x-1} + \frac{2n-1}{x+1} \right] \\ & + \frac{1}{n^2} \sum_{k=1}^{n-1} \frac{\sin^2 \frac{k\pi}{n} \left( 1 - 2x \cos \frac{k\pi}{n} \right)}{\left( x^2 - 2x \cos \frac{k\pi}{n} + 1 \right)^2} + \frac{1}{n^2} \sum_{k=1}^{n-1} \frac{n - \sin^2 \frac{k\pi}{n} - \left( n - \frac{1}{2} \right) x \cos \frac{k\pi}{n}}{x^2 - 2x \cos \frac{k\pi}{n} + 1}. \end{aligned}$$

$$659 \quad a) \frac{\varphi'(x)}{\varphi(x)}; \quad b) \frac{x\varphi'(x) - n\varphi(x)}{\varphi(x)}; \quad c) \frac{[\varphi'(x)]^2 - \varphi(x)\varphi''(x)}{[\varphi(x)]^2}.$$

$$660 \quad a) 9; \quad b) -\frac{\varphi'(2)}{\varphi(2)} + \frac{\varphi'(1)}{\varphi(1)} = -\frac{17}{5}; \quad c) 17.$$

$$661 \quad 0.51x + 2.04.$$

$$662 \quad y = \frac{1}{7} [0.55x^2 + 2.35x + 6.98].$$

663 We have

$$q^n f(p/q) = a_0 p^n + a_1 p^{n-1} q + \dots + a_{n-1} p q^{n-1} + a_n q^n = 0 ;$$

thus

$$\frac{a_0 p^n}{q} = -(a_1 p^{n-1} + \dots + a_{n-1} p q^{n-2} + a_n q^{n-1}),$$

$$\frac{a_n q^n}{p} = -(a_0 p^{n-1} + a_1 p^{n-2} q + \dots + a_{n-1} q^{n-1}).$$

The right member of each of the last two equations is an integer. The numbers  $p, q$  are relatively prime. Therefore,  $q$  divides  $a_0$ , and  $p$  divides  $a_n$ .

Now expand  $f(x)$  in powers of  $x - m$ :

$$f(x) = a_0(x - m)^n + c_1(x - m)^{n-1} + \dots + c_{n-1}(x - m) + c_n.$$

The coefficients  $c_1, c_2, \dots, c_n$  are integers, since  $m$  is an integer. Also,  $c_n = f(m)$ . Now

substitute  $x = p/q$  in the above relationship:

$$f(p/q) = a_0(p - mq)^n + c_1(p - mq)^{n-1}q + \dots + c_{n-1}(p - mq)q^{n-1} + c_n q^n = 0.$$

Thus we can conclude that  $\frac{c_n q^n}{p - mq}$  is an integer.

In view of the fact that  $\frac{p - mq}{q} = \frac{p}{q} - m$  is irreducible, the numbers  $p = mq, q$  are relatively prime. Therefore  $c_n = f(m)$  is divisible by  $p - mq$ , which was to be proved.

664 Detailed reasoning for part a) is as follows.

Possible values of  $p$  are: 1, -1, 2, -2, 7, -7, 14, -14. The only possible value of  $q$  is 1 (we have already taken account of both positive and negative values).

$f(1) = -4$ . Therefore  $p - 1$  must divide 4. This eliminates the values  $p = 1, -2, 7, -7, 14, -14$ . Remaining values to be tested are -1, 2. But  $f(-1) \neq 0$ ;  $f(2) = 0$ . The only rational root is  $x_1 = 2$ .

- b)  $x_1 = -3$ ; c)  $x_1 = -2, x_2 = 3$ ; d)  $x_1 = -3, x_2 = \frac{1}{2}$ ;  
 e)  $\frac{5}{2}, -\frac{3}{4}$ ; f) 1, -2, 3; g)  $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}$ ;  
 h) No rational root; i) -1, -2, -3, +4;  
 k)  $\frac{1}{2}$ ; l)  $x_1 = x_2 = -\frac{1}{2}$ ; m)  $x_1 = x_2 = 1, x_3 = x_4 = -3$ ;  
 n)  $x_1 = 3, x_2 = x_3 = x_4 = x_5 = -1$ ; o)  $x_1 = x_2 = x_3 = 2$ .

665 By problem 663,  $p$  and  $p - q$  are both odd. Therefore  $q$  is even and cannot be 1.

666 By problem 663,  $p = x_1 q = \pm 1$ ,  $p - x_2 q = \pm 1$ . Thus  $(x_2 - x_1)q = \pm 2$ , or 0. The value 0 is impossible, since  $q > 0$ ,  $x_2 \neq x_1$ . For definiteness, suppose  $x_2 > x_1$ ,  $(x_2 - x_1)q = 2$ . This would not be possible unless  $x_2 - x_1 = 2$  or 1. In the first case,  $p$  and  $q$  must be related by  $p = x_1 q + 1$ ,  $q = \frac{2}{x_2 - x_1}$ . Then  $p/q = x_1 + 1/q = (x_1 + x_2)/2$ .

667 Eisenstein's criterion gives the results:

- a)  $p = 2$ ; b)  $p = 3$ ; c)  $p = 3$ , if one expands in powers of  $x - 1$ .

$$\begin{aligned}
 668 \quad X_p(x) &= (x-1)^{p-1} + \frac{p}{1}(x-1)^{p-2} \\
 &\quad + \frac{p(p-1)}{1 \cdot 2}(x-1)^{p-3} + \dots + p.
 \end{aligned}$$

All the coefficients  $C_k = \frac{p(p-1)\dots(p-k+1)}{1 \cdot 2 \dots k}$ ,

$k \leq p-1$ , are divisible by  $p$ , since

$k!C_k = p(p-1)\dots(p-k+1)$  is divisible by  $p$ , but  $k!$  is prime to  $p$ . Thus Eisenstein's criterion applies if  $p$  is prime and we expand in powers of  $x-1$ , as we did.

669 We can use Eisenstein's criterion, if we substitute  $x = y + 1$ , and pay attention to divisibility by  $p$ :

$$X_{p^k}(x) = \varphi(y) = \frac{(y+1)^{p^k} - 1}{(y+1)^{p^{k-1}} - 1}.$$

The highest coefficient of the polynomial  $\varphi$  is 1. Furthermore the constant term of  $\varphi(y)$  is  $\varphi(0) = X_{p^k}(1) = p$ ; this is divisible by  $p$  and not by  $p^2$ . It remains to show that the other coefficients are divisible by  $p$ . The proof will be complete if we show by induction on  $n$  that all the coefficients of the polynomial  $(y+1)^{p^n} - 1$ , except the leading one, are divisible by  $p$ . This is obviously true for  $n = 1$ . Suppose as induction hypothesis that this is true for when the exponent is  $p^{n-1}$ , i.e.

$$(y+1)^{p^{n-1}} = y^{p^{n-1}} + 1 + p w_{n-1}(y), \quad \text{where } w_{n-1}(y)$$

is a polynomial with integer coefficients. Then  
 $(y+1)^{p^n} = (y^{p^{n-1}} + 1 + pw_{n-1}(y))^p = (y^{p^{n-1}} + 1)^p + p\psi(y) =$   
 $= y^{p^n} + 1 + pw_n(y);$   
 where  $\psi(y)$ ,  $w_n(y)$  are polynomials with integral coefficients. Thus,

$$\begin{aligned}\varphi(y) &= \frac{y^{pk} + pw_k(y)}{y^{pk-1} + pw_{k-1}(y)} \\ &= y^{pk-pk-1} + p \frac{w_k(y) - y^{pk-pk-1}w_{k-1}(y)}{y^{pk-1} + pw_{k-1}(y)} = y^{pk-pk-1} + p\chi(y).\end{aligned}$$

The coefficients of the polynomial  $\chi(y)$  are integers since  $\chi(y)$  is the quotient of a division of two polynomials, and the highest coefficient of the divisor is 1. Therefore all the coefficients of the polynomial  $\chi(y)$  except for the leading coefficient are divisible by  $p$ , and Eisenstein's criterion applies.

670 To prove the assertion, assume the contrary:

$$f(x) = \varphi(x)\psi(x).$$

If the factors are non-trivial, both factors are polynomials with integral coefficients and degree exceeding 1, since  $f(x)$  has no rational root, by hypothesis. Set

$$\begin{aligned}\varphi(x) &= b_0x^k + b_1x^{k-1} + \dots + b_k, \\ \psi(x) &= c_0x^m + c_1x^{m-1} + \dots + c_m,\end{aligned}$$

$k \geq 2$ ,  $m \geq 2$ ,  $k + m = n$ . Since  $b_k c_m = a_n$  is divisible by  $p$  and is not divisible by  $p^2$ , we can assume that  $b_k$  is divisible by  $p$ , and  $c_m$  is not divisible by  $p$ .

Let  $i$  be chosen,  $i \geq 0$ , so that  $b_{i+1}, b_{i+2}, \dots$  are divisible by  $p$ ,  $b_i$  is not divisible by  $p$ . Such an  $i$  exists, since  $a_0 = b_0 c_0$  is not divisible by  $p$ . Then  $a_{m+i} = b_i c_m + b_{i+1} c_{m-1} + \dots$  is not divisible by  $p$ , since the first term is not divisible by  $p$  and the remaining terms are divisible by  $p$ . This contradicts the assumption, since  $m + i \geq 2$ .

- 671 Suppose that  $f(x)$  can be written as a product of non-trivial factors, each of which is an irreducible polynomial with integral coefficients. Let  $\varphi(x)$  be an irreducible factor having constant term divisible by  $p$ . There must be such a factor since  $a_n$  is divisible by  $p$ . Let  $\psi(x)$  be the complementary factor:  $f(x) = \varphi(x) \psi(x)$ . The proof now proceeds along the lines of the argument for problem 670. Set

$$\begin{aligned}\varphi(x) &= b_0 x^m + b_1 x^{m-1} + \dots + b_m, \\ \psi(x) &= c_0 x^h + c_1 x^{h-1} + \dots + c_h\end{aligned}$$

and suppose that  $b_{i+1}, b_{i+2}, \dots$  are divisible by  $p$ , but  $b_i$  is prime to  $p$ . Such an  $i$  exists, since  $c_h$  cannot be divisible by  $p$ , otherwise  $a_n = b_m c_h$  would be divisible by  $p^2$ .

Thus  $a_{h+i} = b_i c_h + b_{i+1} c_{h-1} + \dots$  is not divisible by  $p$ , from which the relation  $h + i \leq k$  follows. Therefore  $m \geq m + h + i - k = n + i - k \geq n - k$ .

672 a)  $f(0) = 1, f(1) = -1, f(-1) = -1.$

If  $f(x) = \varphi(x)\psi(x)$ , where the degree of  $\varphi(x) \leq 2$ , then  $\varphi(0) = \pm 1, \varphi(1) = \pm 1, \varphi(-1) = \pm 1$ , that is  $\varphi(x)$  takes one of the sets of tabular values shown below:

$x$	$\varphi(x)$							
$-1$	$1$	$1$	$1$	$1$	$-1$	$-1$	$-1$	$-1$
$0$	$1$	$-1$	$-1$	$1$	$-1$	$1$	$1$	$-1$
$1$	$-1$	$-1$	$1$	$1$	$1$	$1$	$-1$	$-1$

We can consider some of these cases together since the last four cases are the same as the respective negatives of the first four. Moreover the fourth and last cases can be omitted since the corresponding polynomial would be the constant 1. Thus we arrive at the following three possibilities:  
 $\varphi(x) = -(x^2 + x - 1); \varphi(x) = x^2 - x - 1; \varphi(x) = 2x^2 - 1.$

The following factorization can be verified:

$$f(x) = (x^2 + x - 1)(x^2 - x - 1).$$

b) Irreducible; c) Irreducible;

d)  $(x^2 - x - 1)(x^2 - 2).$

673 A reducible first degree polynomial must have a factor of the first degree that has rational coefficients. Thus the original polynomial must have a rational root.

674 If the polynomial  $x^4 + ax^3 + bx^2 + cx + d$  has no rational root, then it must be either irreducible or expressible as the product of two quadratic

polynomials with integral coefficients:

$$x^4 + ax^3 + bx^2 + cx + d = (x^2 + \lambda x + m)(x^2 + \mu x + n).$$

Comparing coefficients we obtain the following relations:

$$\begin{aligned} mn &= d, \\ \lambda + \mu &= a, \\ n\lambda + m\mu &= c. \end{aligned}$$

If  $m \neq n$ , then  $\lambda = \frac{c-am}{n-m} = \frac{cm-am^2}{d-m^2}$ , as the assertion states. If  $m = n$ , then  $c = am$ , and  $c^2 = a^2d$  (see problem 614). This is the exceptional case mentioned in the problem.

675 Suppose the polynomial is reducible. Then

$$\begin{aligned} x^5 + ax^4 + bx^3 + cx^2 + dx + e \\ = (x^2 + \lambda x + m)(x^3 + \lambda'x^2 + \lambda''x + n) \end{aligned}$$

We may assume all explicitly written coefficients to be integers.

Comparing coefficients we obtain the relations:

$$\begin{aligned} mn &= e, \\ n\lambda + m\lambda'' &= d, \\ m + \lambda\lambda' + \lambda'' &= b, \\ n + \lambda\lambda'' + m\lambda' &= c. \end{aligned}$$

Thus

$$\begin{aligned} m\lambda'' - n\lambda' &= d - an, \\ \lambda(m\lambda'' - n\lambda') + m^2\lambda' - n\lambda'' &= cm - bn. \end{aligned}$$

Therefore  $(d - an)\lambda + m^2\lambda' - n\lambda'' = cm - bn$ . Solving this equation together with  $\lambda + \lambda' = a$ ,  $n\lambda + m\lambda'' = d$ , we obtain

$$\lambda = \frac{am^3 - cm^2 - dn + be}{m^3 - n^2 + ae - dm},$$

which is what was to be proved.



- 676 a)  $(x^2 - 2x + 3)(x^2 - x - 3)$ ; b) not factorable;  
 c)  $(x^2 - x - 4)(x^2 + 5x + 3)$ ;  
 d)  $(x^2 - 2x + 2)(x^3 + 3x + 3)$ .

- 677 If the polynomial  $x^4 + px^2 + q$  with rational coefficients is reducible, then it is expressible as a product of a third degree by a first degree polynomial or as a product of two second degree polynomials. But we need not consider the first case, since if  $x_1$  is a root, then  $-x_1$  is also a root and the product  $(x - x_1)(x + x_1)$  is a second degree factor with rational coefficients.

$$\text{Set } x^4 + px^2 + q = (x^2 + \lambda_1 x + \mu_1)(x^2 + \lambda_2 x + \mu_2).$$

Then

$$\begin{aligned}\lambda_1 + \lambda_2 &= 0, \\ \lambda_1 \mu_2 + \lambda_2 \mu_1 &= 0, \\ \mu_1 + \lambda_1 \lambda_2 + \mu_2 &= p, \\ \mu_1 \mu_2 &= q.\end{aligned}$$

If  $\lambda_1 = 0$ , then  $\lambda_2 = 0$ . In this case a necessary and sufficient condition that rational numbers  $\mu_1, \mu_2$  exist is that the discriminant  $p^2 - 4q$  should be the square of a rational number.

The remaining case is  $\lambda_1 \neq 0$ . Then

$$\begin{aligned}\lambda_2 &= -\lambda_1, \mu_2 = \mu_1 \text{ and thus} \\ q &= \mu_1^2, \quad 2\mu_1 - p = \lambda_1^2.\end{aligned}$$

In summary, the polynomial  $x^4 + px^2 + q$  with rational coefficients will be reducible if and only if one of the following two conditions is satisfied.

a)  $p^2 - 4q$  is the square of a rational number.

b)  $q$  is the square of a rational number  $\mu_1$ , and

$2\mu_1 - p$  is the square of a rational number  $\lambda_1$ .

678 Suppose  $x^4 + ax^3 + bx^2 + cx + d = (x^2 + p_1x + q_1)(x^2 + p_2x + q_2)$ .

Then, since  $p_1 + p_2 = a$ , we can write:

$$x^4 + ax^3 + bx^2 + cx + d = \left(x^2 + \frac{1}{2}ax + \frac{\lambda}{2}\right)^2 - \left(\frac{p_1 - p_2}{2}x + \frac{q_1 - q_2}{2}\right)^2,$$

where  $\lambda = q_1 + q_2$ . Thus the auxiliary cubic has

the rational root  $\lambda = q_1 + q_2$ .

679 Suppose  $f(x) = \varphi(x)\psi(x)$  where  $\varphi(x), \psi(x)$  are

polynomials with integral coefficients. Since

$f(a_i) = -1$ , then either  $\varphi(a_i) = 1, \psi(a_i) = -1$ ,

or  $\varphi(a_i) = -1, \psi(a_i) = 1$ . If neither  $\varphi(x)$  or  $\psi(x)$

is constant, then the degree of  $\varphi(x) + \psi(x)$

is less than  $n$ ; thus  $\varphi(x) + \psi(x) \equiv 0$ . This leads

to the contradiction  $f(x) = -[\varphi(x)]^2$ . This is

indeed a contradiction since the coefficient of

the highest power of  $x$  in  $f(x)$  is positive.

680 If  $f(x) = \varphi(x)\psi(x)$ , then  $\varphi(a_i) = \psi(a_i) = \pm 1$ ,

as in the preceding problem. If neither  $\varphi(x)$

or  $\psi(x)$  is constant, then  $\varphi(x) \equiv \psi(x)$

and thus

$$f(x) = [\varphi(x)]^2.$$

This can be true only if the degree  $n$  is even.

Thus the only possibility to consider is

$$(x - a_1)(x - a_2) \dots (x - a_n) + 1 = [\varphi(x)]^2.$$

Assume without loss of generality the coefficient

of the highest power of  $x$  in  $\varphi(x)$  is  $+1$  (and not

-1). Thus

$$\varphi(x) + 1 = (x - a_1)(x - a_3) \dots (x - a_{n-1}).$$

$$\varphi(x) - 1 = (x - a_2)(x - a_4) \dots (x - a_n).$$

(In these last equations the subscripts are a typographical simplification that can be obtained by renumbering the roots  $a_1, a_2, \dots, a_n$ .) Finally we have

$$(x - a_1)(x - a_3) \dots (x - a_{n-1}) - (x - a_2)(x - a_4) \dots (x - a_n) = 2.$$

Suppose  $a_1 > a_3 > \dots > a_{n-1}$ . In the equation last obtained we can set  $x = a_{2k}$ , for  $k = 1, 2, \dots, n/2$ . And obtain:

$$(a_{2k} - a_1)(a_{2k} - a_3) \dots (a_{2k} - a_{n-1}) = 2.$$

Thus the number 2 is factored in  $n/2$  different ways into a product of  $n/2$  integral factors.

Moreover the factors appear in increasing order.

This can occur only if  $n/2 = 2$ ,  $2 = -2(-1) = 1 \cdot 2$ , or also if  $n/2 = 1$ . These two possibilities correspond precisely to the two factorizations mentioned in the problem.

- 681 If the  $n$ -th degree polynomial  $f(x)$  is reducible,  $n = 2m$  or  $n = 2m + 1$ , then  $f(x)$  must have one factor  $\varphi(x)$  of degree not exceeding  $m$ . If  $f(x)$  takes the value  $\pm 1$  for more than  $2m$  integral values of its argument, then  $\varphi(x)$  also takes the value  $\pm 1$  for the same values of the argument. Thus  $\varphi(x)$  must take at least one of the two values  $+1, -1$  more than  $m$  times. This shows that either  $\varphi(x) \equiv +1$  or  $\varphi(x) \equiv -1$ .

- 682 If  $f(x)$  could be expressed as the product  $f(x) = \varphi(x) \psi(x)$  of non-trivial factors, then neither factor could have a real root since  $f(x)$  has no real root. Thus neither factor can change sign as  $x$  traverses the real line, and we may suppose  $\varphi(x) > 0$ ,  $\psi(x) > 0$  for all real  $x$ . Since  $f(a_k) = 1$ , it follows that  $\varphi(a_k) = \psi(a_k) = 1$ ,  $k = 1, 2, \dots, n$ . If the degree of either  $\varphi(x)$  or  $\psi(x)$  were less than  $n$ , then the polynomial would have to be identically 1. Thus each of the factors  $\varphi(x)$ ,  $\psi(x)$  has degree  $n$ .

Thus

$$\varphi(x) = 1 + \alpha(x - a_1) \dots (x - a_n), \psi(x) = 1 + \beta(x - a_1) \dots (x - a_n),$$

where  $\alpha$ ,  $\beta$  are certain integers. Thus we have shown that  $f(x)$  can be represented in the form

$$f(x) = (x - a_1)^2 \dots (x - a_n)^2 + 1 = 1 + (\alpha + \beta)(x - a_1) \dots (x - a_n) + \alpha\beta(x - a_1)^2 \dots (x - a_n)^2.$$

Comparing the coefficients of  $x^{2n}$ ,  $x^n$ , we obtain the relations:  $\alpha\beta = 1$ ,  $\alpha + \beta = 0$ ; but no integers satisfy these relations, and therefore  $f(x)$  is irreducible.

- 683 Suppose that  $f(x)$  takes the value 1 for more than three values of its argument. Then  $f(x) - 1$  must have at least four different integral roots, i.e.

$$f(x) - 1 = (x - a_1)(x - a_2)(x - a_3)(x - a_4)h(x),$$

where  $a_1, a_2, a_3, a_4$  and the coefficients of the polynomial  $h(x)$  are integers. For any particular integral value of  $x$ , the product

$(x-a_1)(x-a_2)(x-a_3)(x-a_4)$  is a product of distinct integers. No more than two of these can be 1 or -1; the other two must always be different from  $\pm 1$ . Therefore the product cannot be a prime number; in particular cannot be -2. Thus,  $f(x) - 1 \neq -2$ , for any integral value of  $x$ . Therefore  $f(x) \neq -1$ .

- 684 Suppose that  $f(x) = \varphi(x)\psi(x)$ . One of the polynomials  $\varphi(x)$  has degree  $\leq n/2$ , and takes one of the values  $\pm 1$  for more than  $n/2$  integral values of  $x$ . Since  $n/2 \geq 6$ ,  $\varphi(x)$  or  $-\varphi(x)$  takes the value 1 more than three times, and according to problem 683, cannot take the value of -1 at all. Thus, either  $\varphi(x)$  or  $-\varphi(x)$  takes the value +1 more than  $n/2$  times. It must therefore be identically 1 and  $f(x)$  is irreducible. For  $n \geq 8$ , the assertion can be established by a different argument.

- 685 Suppose

$$a[\varphi(x)]^2 + b\varphi(x) + 1 = \psi(x)\omega(x).$$

One of the polynomials has degree  $\leq n$ ;  $\psi(x)$  takes the value  $\pm 1$  for  $x = a_1, a_2, \dots, a_n$ , and since  $n \geq 7$ , all these values of  $\psi(x)$  must be of the same sign. Therefore,

$$\psi(x) = \pm 1 + \alpha(x-a_1)(x-a_2)\dots(x-a_n) = \pm 1 + \alpha\varphi(x).$$

If  $\alpha \neq 0$ , then  $\omega(x)$  must also have degree  $n$ , and  $\omega(x) = \pm 1 + \beta\varphi(x)$ . But the equality

$$a[\varphi(x)]^2 + b\varphi(x) + 1 = [\pm 1 + \alpha\varphi(x)][\pm 1 + \beta\varphi(x)]$$

cannot subsist, since by hypothesis the polynomial  $ax^2 + bx + 1$  is irreducible.

$$686 \quad a) \quad f(x) = a_0 x^n \left( 1 + \frac{a_1}{a_0 x} + \dots + \frac{a_n}{a_0 x^n} \right).$$

Set  $\left| \frac{a_k}{a_0} \right| = A$ . Then for  $|x| > 1$ :

$$|f(x)| \geq |a_0 x^n| \left[ 1 - \frac{A}{|x|-1} \right] = |a_0 x^n| \frac{|x|-1-A}{|x|-1}; \text{ this is}$$

positive for  $|x| > 1 + A$ .

$$b) \quad \frac{1}{\rho^n} f(x) = a_0 \left( \frac{x}{\rho} \right)^n + \frac{a_1}{\rho} \left( \frac{x}{\rho} \right)^{n-1} + \frac{a_2}{\rho^2} \left( \frac{x}{\rho} \right)^{n-2} + \dots + \frac{a_n}{\rho^n}.$$

By a) we find that all the roots satisfy the relations

$$\frac{|x|}{\rho} \leq 1 + \max \left| \frac{a_k}{a_0 \rho^k} \right|, \quad \text{thus} \quad |x| \leq \rho + \max \left| \frac{a_k}{a_0 \rho^{k-1}} \right|.$$

c) Set  $\rho = \max \sqrt[k]{\left| \frac{a_k}{a_0} \right|}$ . Then

$$\left| \frac{a_k}{a_0} \right| \leq \rho^k, \quad \left| \frac{a_k}{a_0 \rho^{k-1}} \right| \leq \rho, \quad \max \left| \frac{a_k}{a_0 \rho^{k-1}} \right| \leq \rho.$$

Therefore the moduli of all the roots are bounded by the number

$$\rho + \rho = 2\rho = 2 \max \sqrt[k]{\left| \frac{a_k}{a_0} \right|}.$$

d) Set  $\rho = \max \sqrt[k-1]{\left| \frac{a_k}{a_1} \right|}$ . Then

$$|a_k| \leq |a_1| \rho^{k-1}, \quad \left| \frac{a_k}{a_0 \rho^{k-1}} \right| \leq \left| \frac{a_1}{a_0} \right|.$$

Therefore no root has modulus exceeding

$$\rho + \left| \frac{a_1}{a_0} \right| = \left| \frac{a_1}{a_0} \right| + \max \sqrt[k-1]{\left| \frac{a_k}{a_1} \right|}.$$

687 Set  $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$ ,  
 $\varphi(x) = b_0x^n - b_1x^{n-1} - \dots - b_n$ ;

$0 < b_0 \leq |a_0|$ ,  $b_1 \geq |a_1|$ , ...,  $b_n \geq |a_n|$ . Obviously,

$$|f(x)| \geq \varphi(|x|).$$

$$\text{Thus, } \varphi(x) = b_0x^n \left(1 - \frac{b_1}{b_0x} - \frac{b_2}{b_0x^2} - \dots - \frac{b_n}{b_0x^n}\right).$$

The factor in parentheses increases from  $-\infty$  to 1 as  $x$  varies from 0 to  $\infty$ .

Thus  $\varphi(x)$  has a single positive root  $\xi$  and  $\varphi(x) > 0$  for  $x > \xi$ . Thus, for  $|x| > \xi$ , the relation  $|f(x)| \geq \varphi(|x|) > 0$ , holds, and therefore no root of  $f(x)$  exceeds  $\xi$  in modulus.

688 a) Set  $A = \max \left| \frac{a_k}{a_0} \right|$ . It is obvious that

$$|f(x)| \geq |a_0x^n| \left(1 - \frac{A}{|x|^r} - \frac{A}{|x|^{r+1}} - \dots - \frac{A}{|x|^n}\right).$$

Thus, for  $|x| > 1$ , we have

$$\begin{aligned} |f(x)| &> |a_0x^n| \left(1 - \frac{A}{|x|^{r-1}(|x|-1)}\right) \\ &= \frac{|a_0x^{n-r+1}|}{|x|-1} [|x|^{r-1}(|x|-1) - A] > \frac{|a_0x|^{n-r+1}}{|x|-1} [(|x|-1)^r - A]. \end{aligned}$$

For  $|x| > 1 + \sqrt[r]{A}$ ,  $|f(x)| > 0$ .

b)  $\frac{1}{\rho^n} f(x) = a_0 \left(\frac{x}{\rho}\right)^n + \frac{a_r}{\rho^r} \left(\frac{x}{\rho}\right)^{n-r} + \dots + \frac{a_n}{\rho^n}$ .

By a) all roots of  $f(x)$  satisfy the relation

$$\left|\frac{x}{\rho}\right| < 1 + \sqrt[r]{\max \left| \frac{a_k}{a_0 \rho^k} \right|}, \text{ so that } |x| < \rho + \sqrt[r]{\max \left| \frac{a_k}{a_0 \rho^{k-r}} \right|}.$$

c) Set  $\rho = \max \sqrt[k-1]{\left| \frac{a_k}{a_r} \right|}$ . Then  $|a_k| \leq |a_r| \rho^{k-r}$ ,

and no root of the polynomial exceeds

$$\sqrt[r]{\left|\frac{a_r}{a_0}\right|} + \rho = \sqrt[r]{\left|\frac{a_r}{a_0}\right|} + \max_k \sqrt[k-r]{\left|\frac{a_k}{a_r}\right|}$$

in modulus.

- 689 The assertion is obvious for the negative roots of the polynomial. For definiteness suppose  $a_0 > 0$ , and set  $\varphi(x) = a_0 x^n - b_1 x^{n-1} - b_2 x^{n-2} - \dots - b_n$ , where  $b_k = 0$  for  $a_k > 0$ ,  $b_k = -a_k$  for  $a_k < 0$ . Then if  $x$  is positive, we have

$$f(x) \geq \varphi(x).$$

Thus  $\varphi(x)$  has a single non-negative root  $\xi$  (see problem 687) and  $\varphi(x) > 0$  for  $x > \xi$ . Thus, for  $x > \xi$ ,  $f(x) \geq \varphi(x) > 0$ .

- 690 Follows directly from problems 688, 689, 686 c).
- 692 Expand  $f(x)$  in powers of  $x - a$ . For  $x \leq a$ , we have:

$$f(x) = f(a) + \frac{f'(a)}{1} (x-a) + \frac{f''(a)}{1 \cdot 2} (x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!} (x-a)^n > 0.$$

- 693 We use problems 690, 692 to obtain an upper bound for the roots. A lower bound can be obtained by considering  $f(-x)$ :
- a)  $0 < x_i < 3$ ; b)  $0 < x_i < 1$ ; c)  $-11 < x_i < 11$ ; d)  $-6 < x_i < 2$ .
- 694 a)  $f = x^3 - 3x - 1$ ,  $f_1 = x^2 - 1$ ,  $f_2 = 2x + 1$ ,  $f_3 = +1$ .

Three real roots in the intervals

$$(-2, -1), (-1, 0), (1, 2).$$

$$b) f = x^3 + x^2 - 2x - 1, f_1 = 3x^2 + 2x - 2, f_2 = 2x + 1, f_3 = +1.$$

Three real roots in the intervals

$$(-2, -1), (-1, 0), (1, 2).$$



c)  $f = x^3 - 7x + 7$ ,  $f_1 = 3x^2 - 7$ ,  $f_2 = 2x - 3$ ,  $f_3 = +1$ .

Three real roots in the intervals

$(-4, -3)$ ,  $(1, 3/2)$ ,  $(3/2, 2)$ .

d)  $f = x^3 - x + 5$ ,  $f_1 = 3x^2 - 1$ ,  $f_2 = 2x - 15$ ,  $f_3 = -1$ .

One real root in the interval  $(-2, -1)$ .

e)  $f = x^3 + 3x - 5$ ,  $f_1 = x^2 + 1$ .

One real root in the interval  $(1, 2)$ .

695 a)  $f = x^4 - 12x^2 - 16x - 4$ ,  $f_1 = x^3 - 6x - 4$ ,

$f_2 = 3x^2 + 6x + 2$ ,  $f_3 = x + 1$ ,  $f_4 = 1$ .

Four real roots in the intervals

$(-3, -2)$ ,  $(-2, -1)$ ,  $(-1, 0)$ ,  $(4, 5)$ .

b)  $f = x^4 - x - 1$ ,  $f_1 = 4x^3 - 1$ ,  $f_2 = 3x + 4$ ,  $f_3 = +1$ .

Two real roots in the intervals

$(-1, 0)$ ,  $(1, 2)$ .

c)  $f = 2x^4 - 8x^3 + 8x^2 - 1$ ,  $f_1 = x^3 - 3x^2 + 2x$ ,  $f_2 = 2x^2 - 4x + 1$ ,  
 $f_3 = x - 1$ ,  $f_4 = 1$ .

Four real roots in the intervals  $(-1, 0)$ ,

$(0, 1)$ ,  $(1, 2)$ ,  $(2, 3)$ .

d)  $f = x^4 + x^2 - 1$ ,  $f_1 = 2x^3 + x$ ,  $f_2 = -x^2 + 2$ ,  $f_3 = -x$ ,  
 $f_4 = -1$ .

Two real roots in the intervals  $(-1, 0)$ ,

$(0, 1)$ .

e)  $f = x^4 + 4x^3 - 12x + 9$ ,  $f_1 = x^3 + 3x^2 - 3$ ,  $f_2 = x^2 + 3x - 4$ ,  
 $f_3 = -4x + 3$ ,  $f_4 = 1$ .

No real roots.

696 a)  $f = x^4 - 2x^3 - 4x^2 + 5x + 5$ ,  $f_1 = 4x^3 - 6x^2 - 8x + 5$ ,

$f_2 = 22x^2 - 22x - 45$ ,  $f_3 = 2x - 1$ ,  $f_4 = 1$ .

Four real roots in the intervals  $(1, 2)$ ,

$(2, 3)$ ,  $(-1, 0)$ ,  $(-2, -1)$ .

b)  $f = x^4 - 2x^3 + x^2 - 2x + 1$ ,  $f_1 = 2x^3 - 3x^2 + x - 1$ ,  
 $f_2 = x^2 + 5x - 3$ ,  $f_3 = -9x + 5$ ,  $f_4 = -1$ .

Two real roots in the intervals  $(0, 1)$ ,  $(1, 2)$ .

c)  $f = x^4 - 2x^3 - 3x^2 + 2x + 1$ ,  $f_1 = 2x^3 - 3x^2 - 3x + 1$ ,  
 $f_2 = 9x^2 - 3x - 5$ ,  $f_3 = 9x + 1$ ,  $f_4 = +1$ .

Four real roots in the intervals  $(-2, -1)$ ,  
 $(-1, 0)$ ,  $(0, 1)$ ,  $(2, 3)$ .

d)  $f = x^4 - x^3 + x^2 - x - 1$ ,  $f_1 = 4x^3 - 3x^2 + 2x - 1$ ,  
 $f_2 = -5x^2 + 10x + 17$ ,  $f_3 = -8x - 5$ ,  $f_4 = -1$ .

Two real roots in the intervals  $(1, 2)$ ,  $(-1, 0)$ .

e)  $f = x^4 - 4x^3 - 4x^2 + 4x + 1$ ,  $f_1 = x^3 - 3x^2 - 2x + 1$ ,  
 $f_2 = 5x^2 - x - 2$ ,  $f_3 = 18x + 1$ ,  $f_4 = +1$ .

Four real roots in the intervals  $(-2, -1)$ ,  
 $(-1, 0)$ ,  $(0, 1)$ ,  $(4, 5)$ .

697 a)  $f = x^4 - 2x^3 - 7x^2 + 8x + 1$ ,  $f_1 = 2x^3 - 3x^2 - 7x + 4$ ,  
 $f_2 = 17x^2 - 17x - 8$ ,  $f_3 = 2x - 1$ ,  $f_4 = 1$ .

Four real roots in the intervals  $(-3, -2)$ ,  
 $(-1, 0)$ ,  $(1, 2)$ ,  $(3, 4)$ .

b)  $f = x^4 - 4x^2 + x + 1$ ,  $f_1 = 4x^3 - 8x + 1$ ,  $f_2 = 8x^2 - 3x - 4$ ,  
 $f_3 = 87x - 28$ ,  $f_4 = +1$ .

Four real roots in the intervals  $(-3, -2)$ ,  
 $(-1, 0)$ ,  $(0, 1)$ ,  $(1, 2)$ .

c)  $f = x^4 - x^3 - x^2 - x + 1$ ,  $f_1 = 4x^3 - 3x^2 - 2x - 1$ ,  
 $f_2 = 11x^2 + 14x - 15$ ,  $f_3 = -8x + 7$ ,  $f_4 = -1$ .

Two real roots in the intervals  $(0, 1)$ ,  $(1, 2)$ .

d)  $f = x^4 - 4x^3 + 8x^2 - 12x + 8$ ,  $f_1 = x^3 - 3x^2 + 4x - 3$ ,  
 $f_2 = -x^2 + 5x - 5$ ,  $f_3 = -9x + 13$ ,  $f_4 = -1$ .

Two real roots  $x_1 = 2$ ,  $1 < x_2 < 2$ .

e)  $f = x^4 - x^3 - 2x + 1$ ,  $f_1 = 4x^3 - 3x^2 - 2$ ,  $f_2 = 3x^2 + 24x - 14$ ,  
 $f_3 = -56x + 31$ ,  $f_4 = -1$ .

Two real roots in the intervals  $(0, 1)$ ,  $(1, 2)$ .

698 a)  $f = x^4 - 6x^2 - 4x + 2$ ,  $f_1 = x^3 - 3x - 1$ ,  
 $f_2 = 3x^2 + 3x - 2$ ,  $f_3 = 4x + 5$ ,  $f_4 = 1$ .

Four real roots in the intervals  $(-2, -3/2)$ ,  
 $(-3/2, -1)$ ,  $(0, 1)$ ,  $(2, 3)$ .

b)  $f = 4x^4 - 12x^2 + 8x - 1$ ,  $f_1 = 2x^3 - 3x + 1$ ,  
 $f_2 = 6x^2 - 6x + 1$ ,  $f_3 = 2x - 1$ ,  $f_4 = 1$ .

Four real roots in the intervals  $(-3, -2)$ ,  
 $(0, 1/2)$ ,  $(1/2, 1)$ ,  $(1, 2)$ .

c)  $f = 3x^4 + 12x^3 + 9x^2 - 1$ ,  $f_1 = 2x^3 + 6x^2 = 3x$ ,  
 $f_2 = 9x^2 + 9x + 2$ ,  $f_3 = 13x + 8$ ,  $f_4 = 1$ .

Four real roots in the intervals  $(-4, -3)$ ,  
 $(-1, -2/3)$ ,  $(-2/3, -1/2)$ ,  $(0, 1)$ .

d)  $f = x^4 - x^3 - 4x^2 + 4x + 1$ ,  $f_1 = 4x^3 - 3x^2 - 8x + 4$ ,  
 $f_2 = 7x^2 - 8x - 4$ ,  $f_3 = 4x - 5$ ,  $f_4 = 1$ .

Four real roots in the intervals  $(1, 3/2)$ ,  
 $(3/2, 2)$ ,  $(-2, -1)$ ,  $(-1, 0)$ .

e)  $f = 9x^4 - 126x^2 - 252x - 140$ ,  $f_1 = x^3 - 7x - 7$ ,  
 $f_2 = 9x^2 + 27x + 20$ ,  $f_3 = 2x + 3$ ,  $f_4 = 1$ .

Four real roots in the intervals  $(4, 5)$ ,  
 $(-4/3, -1)$ ,  $(-5/3, -4/3)$ ,  $(-2, -5/3)$ .

699 a)  $f = 2x^5 - 10x^3 + 10x - 3$ ,  $f_1 = x^4 - 3x^2 + 1$ ,  
 $f_2 = 4x^3 - 8x + 3$ ,  $f_3 = 4x^2 + 3x - 4$ ,  $f_4 = x$ ,  $f_5 = 1$ .

Five real roots in the intervals  $(-2, -3/2)$ ,  
 $(-3/2, -1)$ ,  $(0, 1/2)$ ,  $(1/2, 1)$ ,  $(1, 2)$ .

b)  $f = x^6 - 3x^5 - 3x^4 + 11x^3 - 3x^2 - 3x + 1$ ,  $f_1 = 2x^5 - 5x^4$   
 $- 4x^3 + 11x^2 - 2x - 1$ ,  $f_2 = 3x^4 - 6x^3 - x^2 + 4x - 1$ ,  $f_3 = 4x^3$   
 $- 6x^2 + 1$ ,  $f_4 = 26x^2 - 26x + 5$ ,  $f_5 = 2x - 1$ ,  $f_6 = 1$ .

Six real roots in the intervals  $(-2, -1)$ ,  
 $(-1, 0)$ ,  $(0, 1/2)$ ,  $(1/2, 1)$ ,  $(1, 2)$ ,  $(2, 3)$ .

c)  $f = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$ ,  $f_1 = 5x^4 + 4x^3 - 12x^2 - 6x + 3$ ,  $f_2 = 4x^3 + 3x^2 - 6x - 2$ ,  $f_3 = 3x^2 + 2x - 2$ ,  $f_4 = 2x + 1$ ,  $f_5 = 1$ .

Five real roots in the intervals  $(-2, -3/2)$ ,  $(-3/2, -1)$ ,  $(-1, 0)$ ,  $(0, 1)$ ,  $(1, 2)$ .

d)  $f = x^5 - 5x^3 - 10x^2 + 2$ ,  $f_1 = x^4 - 3x^2 - 4x$ ,  $f_2 = x^3 + 3x^2 - 1$ ,  $f_3 = -2x^2 + x + 1$ ,  $f_4 = -3x - 1$ ,  $f_5 = -1$ .

Three real roots in the intervals  $(-1, 0)$ ,  $(0, 1)$ ,  $(2, 3)$ .

700 a)  $f = x^4 + 4x^2 - 1$ ,  $f_1 = x$ ,  $f_2 = 1$ .

Two real roots in the intervals  $(-1, 0)$ ,  $(0, 1)$ .

b)  $f = x^4 - 2x^3 + 3x^2 - 9x + 1$ ,  $f_1 = 2x - 3$ ,  $f_2 = 1$ .

Two real roots in the intervals  $(0, 1)$ ,  $(2, 3)$ .

c)  $f = x^4 - 2x^3 + 2x^2 - 6x + 1$ ,  $f_1 = 2x - 3$ ,  $f_2 = 1$ .

Two real roots in the intervals  $(0, 1)$ ,  $(2, 3)$ .

d)  $f = x^5 + 5x^4 + 10x^2 - 5x - 3$ ,  $f_1 = x^2 + 4x - 1$ ,  $f_2 = 5x - 1$ ,  $f_3 = 1$ .

Three real roots in the intervals  $(0, 1)$ ,  $(-1, 0)$ ,  $(-6, -5)$ .

701 The Sturm sequence consists of the polynomials

$$x^3 + px + q, 3x^2 + p, -2px - 3q, -4p^3 - 27q^2.$$

If  $-4p^3 - 27q^2 > 0$ , then  $p < 0$ . In each

polynomial in the Sturm sequence the highest

coefficient is positive and thus all the roots of

$x^3 + px + q$  are real. If  $-4p^3 - 27q^2 < 0$ , then

the sign of  $p$  does not matter; the Sturm

sequence has two sign changes for  $x = -\infty$ ; one

change for  $x = +\infty$ . In this case  $x^3 + px + q$  has

one real root.

## 702 The polynomials

$$x^n + px + q, \quad nx^{n-1} + p, \quad -(n-1)px - nq, \quad -p - n\left(\frac{-nq}{(n-1)p}\right)^{n-1}$$

form the Sturm sequence.

If  $n$  is odd, the sign of the last polynomial is the same as the sign of  $\Delta = -(n-1)^{n-1}p^n - n^nq^{n-1}$ . If  $\Delta > 0$ , then we must have  $p < 0$ . In this case the polynomial has three real roots. If  $\Delta < 0$ , then the sign of  $p$  does not matter, and the polynomial has one real root.

If  $n$  is even, the sign of the last number in the Sturm sequence is the same as the sign of  $-p\Delta$ , where  $\Delta = (n-1)^{n-1}p^n - n^nq^{n-1}$ .

The following table shows the succession of signs for the four possible combinations of values of  $p, \Delta$ :

		$f$	$f_1$	$f_2$	$f_3$
1. $p > 0, \Delta > 0$	$-\infty$	+	+	+	+
	$+\infty$	+	+	+	+
2. $p < 0, \Delta > 0$	$-\infty$	+	+	+	+
	$+\infty$	+	+	+	+
3. $p > 0, \Delta < 0$	$-\infty$	+	+	+	+
	$+\infty$	+	+	+	+
4. $p < 0, \Delta < 0$	$-\infty$	+	+	+	+
	$+\infty$	+	+	+	+

Examination of this table shows that if  $\Delta > 0$ , the polynomial has two real roots; if  $\Delta < 0$ , the polynomial has no real root.

- 703 The Sturm sequence consists of the polynomials  
 $f = x^5 - 5ax^3 + 5a^2x + 2b$ ,  $f_1 = x^4 - 3ax^2 + a^2$ ,  
 $f_2 = ax^3 - 2a^2x - b$ ,  $f_3 = a(a^2x^2 - bx - a^3)$ ,  $f_4 = a(a^5 - b^2)x$ ,  
 $f_5 = 1$ .

If  $\Delta = a^5 - b^2 > 0$ , then  $a > 0$ , and every

Sturm polynomial has positive leading coefficient.

In this case all five roots of the polynomial  $f$  are real. If  $\Delta < 0$ , then the following table shows the possibilities depending on the sign of  $a$ .

	$f$	$f_1$	$f_2$	$f_3$	$f_4$	$f_5$
$a > 0$	$-\infty$	$-$	$+$	$-$	$+$	$+$
	$+\infty$	$+$	$+$	$+$	$-$	$+$
$a < 0$	$-\infty$	$-$	$+$	$-$	$-$	$+$
	$+\infty$	$+$	$+$	$-$	$+$	$+$

Thus the polynomial  $f$  has one real root if  $\Delta < 0$ .

- 704 Let  $f_\lambda, f_{\lambda+1}$  be two consecutive polynomials of a "complete" Sturm sequence. If the leading coefficient in every polynomial has the same sign, then there are no sign changes for  $x = +\infty$ , and there is a sign change for  $x = -\infty$ , since the degrees are of different parity. If the highest coefficients have opposite sign, then for  $x = +\infty$ ,  $f_\lambda, f_{\lambda+1}$  exhibit a sign change but for  $x = -\infty$ , they do not. Thus if  $v_1, v_2$  is the number of sign changes in the Sturm sequence for  $-\infty, +\infty$ , then  $v_1 + v_2 = n$ . On the other hand  $v_1 - v_2$  is the number  $N$  of real roots of the polynomial. Therefore  $v_2 = (n-N)/2$ , which was to be shown.

- 705 The proof runs like the proof of Sturm's theorem except that the number of sign changes when  $x$  goes through a root of the original polynomial is not necessarily a unit.
- 706 The sequence of polynomials given in the problem is the Sturm series for the interval  $x_0 \leq x < +\infty$  and satisfies the conditions of problem 705 on the interval  $-\infty < x \leq x_0$ . Therefore the number of roots of  $f$  in the interval  $(x_0, \infty)$  is  $v(x_0) - v(+\infty)$ ; the number of roots of  $f$  in the interval  $(-\infty, x_0)$  is  $v(x_0) - v(-\infty)$ , where  $v$  is the number of changes of sign of the polynomial over the interval in question.

In general the number of real roots is equal to

$$2v(x_0) - v(+\infty) - v(-\infty).$$

- 707 We use Euler's theorem in the form

$$P_n = (-1)^n e^{\frac{x^2}{2}} \frac{d^n e^{-\frac{x^2}{2}}}{dx^n} = (-1)^n e^{\frac{x^2}{2}} \frac{d^{n-1} \left( -x e^{-\frac{x^2}{2}} \right)}{dx^{n-1}}$$

to obtain

$$P_n (-1)^{n-1} e^{\frac{x^2}{2}} \left( x \frac{d^{n-1} e^{-\frac{x^2}{2}}}{dx^{n-1}} + (n-1) \frac{d^{n-2} e^{-\frac{x^2}{2}}}{dx^{n-2}} \right)$$

Thus

$$P_n = x P_{n-1} = (n-1) P_{n-2}.$$

On the other hand we can differentiate the equation that defines  $P_{n-1}$ , and obtain

$$P'_{n-1} = (-1)^{n-1} x e^{\frac{x^2}{2}} \frac{d^{n-1} e^{-\frac{x^2}{2}}}{dx^{n-1}} + (-1)^{n-1} e^{\frac{x^2}{2}} \frac{d^n e^{-\frac{x^2}{2}}}{dx^n},$$

so that

$$P'_{n-1} = xP_{n-1} - P.$$

Comparing this result with the previous formula, we obtain  $P'_{n-1} = (n-1)P_{n-2}$  and therefore  $P'_n = nP_{n-1}$ .

These formulas show that the sequence  $P_n, P_{n-1}, \dots, P_1, P_0 = 1$  is a Sturm sequence for the polynomial  $P_n$ , since  $P_{n-1}$  is a positive constant multiplied by  $P'_n$ , and  $P_{\lambda-1}$  is the negative of the remainder when  $P_{\lambda+1}$  is divided by  $P_\lambda$  (except for a positive factor).

Every polynomial  $P_n$  has +1 for highest leading coefficient. Therefore all the roots of  $P_n$  are real.

708 By differentiating the equation defining  $P_n$ , we obtain

$$P'_n = (-1)^n e^x \frac{d^n (x^n e^{-x})}{dx^n} + (-1)^n e^x \frac{d^n (nx^{n-1} e^{-x} - x^n e^{-x})}{dx^n};$$

thus

$$P'_n = (-1)^n n e^x \frac{d^n x^{n-1} e^{-x}}{dx^n}.$$

From this we obtain

$$\begin{aligned} P_n &= (-1)^n e^x \frac{d^n (x \cdot x^{n-1} e^{-x})}{dx^n} \\ &= (-1)^n e^x \left[ x \frac{d^n x^{n-1} e^{-x}}{dx^n} + n \frac{d^{n-1} x^{n-1} e^{-x}}{dx^{n-1}} \right] = \frac{x}{n} P'_n - n P_{n-1}; \end{aligned}$$

$$\text{thus } xP'_n = nP_n + n^2 P_{n-1}.$$

On the other hand,

$$P'_n = (-1)^n n e^x \frac{d^{n-1} [(n-1) x^{n-2} e^{-x} - x^{n-1} e^{-x}]}{dx^{n-1}},$$

$$\text{thus } P'_n = -nP'_{n-1} + nP_{n-1}. \quad \text{Multiplying by } x \text{ and}$$



replacing  $xP'_n$ ,  $xP'_{n-1}$  by their values in terms of  $P_n$ ,  $P_{n-1}$ ,  $P_{n-2}$ , we obtain

$$P_n = (x - 2n + 1)P_{n-1} - (n-1)^2 P_{n-2}.$$

This relation shows that the polynomials  $P_n$  cannot be 0 simultaneously, and if  $P_{n-1} = 0$ , then  $P_n$ ,  $P_{n-2}$  have opposite signs. Thus from

$$\frac{P_{n-1}}{P_n} = -\frac{1}{n} + \frac{xP'_n}{n^2 P_n}$$

we see that  $P_{n-1}/P_n$  changes sign from minus to plus when  $x$  goes through a root of  $P_n$ . Therefore the sequence  $P_n, P_{n-1}, \dots, P_1, P_0 = 1$  is a Sturm sequence for  $P_n$  on the interval  $(0, \infty)$ . The highest coefficient of every polynomial  $P_n$  is 1.  $P_n(0) = (-1)^n n!$ . Therefore  $v(0) - v(+\infty) = n$ , that is  $P_n$  has  $n$  positive roots.

- 709 From the relations  $E'_n = E_{n-1}$ ,  $E_n = E_{n-1} - (-x^n/n!)$ , it follows that the polynomials  $E_n$ ,  $E_{n-1}$  and  $-x^n/n!$  form a Sturm series for  $E_n$  on the interval  $-\infty, -\epsilon$ , if  $\epsilon$  is a sufficiently small positive number. The sign changes on this interval are given by the following table:

$$\begin{array}{c|cc} -\infty & (-1)^n & (-1)^{n-1} & (-1)^{n-1} \\ -\epsilon & + & + & (-1)^{n-1} \end{array}$$

Therefore if  $n$  is even the polynomial  $E_n$  has no negative root; if  $n$  is odd the polynomial  $E_n$  has one negative root. Finally it is obvious that  $E_n(x) > 0$  for  $x \geq 0$ .

- 710 The first step is to apply Euler's formula to the identity

$$\frac{d^{n+1} \left( x^2 e^{\frac{1}{x}} \right)}{dx^{n+1}} = \frac{d^n \left[ (2x-1) e^{\frac{1}{x}} \right]}{dx^n}.$$

We obtain

$$\begin{aligned} x^2 \frac{d^{n+1} e^{\frac{1}{x}}}{dx^{n+1}} + 2(n+1)x \frac{d^n e^{\frac{1}{x}}}{dx^n} + (n+1)n \frac{d^{n-1} e^{\frac{1}{x}}}{dx^{n-1}} &= \\ &= (2x-1) \frac{d^n e^{\frac{1}{x}}}{dx^n} + 2n \frac{d^{n-1} e^{\frac{1}{x}}}{dx^{n-1}}, \end{aligned}$$

Thus  $P_n = (2nx+1)P_{n-1} - n(n-1)P_{n-2}x^2$ . On the other hand from the differential equation defining  $P_{n-1}$  we obtain

$$P_n = (2nx+1)P_{n-1} - x^2 P'_{n-1}.$$

This result shows that  $P'_{n-1} = n(n-1)P_{n-2}$

and therefore  $P'_n = (n+1)nP_{n-1}$ . Putting these facts together we see that the series

$P_n, P_{n-1}, P_{n-2}, \dots, P_0 = 1$  is a Sturm series for  $P_n$ . Every polynomial  $P_n$  has positive leading coefficient. Therefore all the roots of  $P_n$  are real.

- 711 Two forms of the relation

$$\frac{d^n \left( \frac{x^2}{x^2+1} \right)}{dx^n} = - \frac{d^n \left( \frac{1}{x^2+1} \right)}{dx^n},$$

give

$$P_n - 2xP_{n-1} + (x^2+1)P_{n-2} = 0.$$

From the formula defining  $P_{n-1}$ , we obtain by differentiation  $P_n = 2xP_{n-1} - \frac{x^2+1}{n}P'_{n-1}$  ;

thus  $P'_{n-1} = nP_{n-2}$ . Therefore  $P'_n = (n+1)P_{n-1}$ .

From the computations just performed, we see that  $P_n, P_{n-1}, \dots, P_0 = 1$  form a Sturm sequence for  $P_n$ . Every polynomial in this sequence has positive leading coefficient. Thus all roots of  $P_n$  are real.

The problem admits a direct solution. One has

$$\frac{1}{x^2+1} = \frac{1}{2i} \left( \frac{1}{x-i} + \frac{1}{x+i} \right).$$

From this one computes directly that

$$P_n(x) = \frac{1}{2i} [(x+i)^{n+1} - (x-i)^{n+1}].$$

It is easy to see that the roots of  $P_n$  are  $\cot\{k\pi/(n+1)\}$ ,  $k = 1, 2, \dots, n$ .

712 We apply Euler's formula to the identity

$$\frac{d^n \frac{x^2+1}{\sqrt{x^2+1}}}{dx^n} = \frac{d^{n-1} \frac{x}{\sqrt{x^2+1}}}{dx^{n-1}},$$

and obtain

$$P_n - (2n-1)xP_{n-1} + (n-1)^2(x^2+1)P_{n-2} = 0.$$

The equation defining  $P_{n-1}$  gives, on differentiation, the relation

$$P_n - (2n-1)xP_{n-1} + (x^2+1)P'_{n-1} = 0.$$

Thus

$$P'_{n-1} = (n-1)^2 P_{n-2}, \quad P'_n = n^2 P_{n-1}.$$

From the relations already obtained, it follows that  $P_n, P_{n-1}, \dots, P_0 = 1$  form a Sturm sequence.

Each polynomial  $P_n$  has positive leading coefficient. Hence all roots of  $P_n$  are positive.

- 713 The function  $F(x)$ ,  $F'(x)$ ,  $[f'(x)]^2$  form a Sturm sequence for  $F$ . The leading coefficients of this sequence:  $3a_0^2$ ,  $12a_0^2$ ,  $9a_0^2$  are positive. Therefore the number of changes in sign for  $x = -\infty$  differs from the number for  $x = +\infty$  by 2.

If  $f$  has a double root, then  $F$  has one triple root and one simple one. If  $f$  has a triple root then  $F$  has a four-fold root.

- 714 If some polynomial of a Sturm sequence has a multiple root  $x_0$  or a complex root  $\alpha$ , then this polynomial can be replaced by a polynomial of lower degree obtained by dividing out the positive value  $(x - x_0)^2$  or  $(x - \alpha)(x - \bar{\alpha})$ . The succeeding polynomials are obtained in the usual way by applying the Euclidean algorithm and reversing the sign of the remainder. When this is done the number of changes of sign for  $x = -\infty$  becomes  $\leq n - 2$ , where  $n$  is the degree of the original polynomial. Therefore the number of real roots must be  $\leq n - 2$ .

- 715 Let  $F(x) = (x^2 - 1)^n$ .  $F(x)$  has  $-1, +1$  as  $n$ -fold roots.  $F'(x)$  has  $-1, +1$  as  $n - 1$ -fold roots. Moreover by Rolle's theorem there is an additional root in the interval  $-1, +1$ .  $F''(x)$  has  $-1, +1$  as  $n - 2$ -fold roots, and two additional roots in the open interval  $-1, +1$ ; etc.  $F^{(n)}(x) = P_n(x)$  has  $n$  roots in the open interval  $-1, 1$ .

716 Let  $x_1, \dots, x_k$  be the distinct roots of  $f(x)$ ,  
 $x_1 < x_2 < x_3 < \dots < x_k$ ; let their multiplicities  
 be respectively  $\alpha_1, \alpha_2, \dots, \alpha_k$ . The function  
 $\varphi(x) = f'(x)/f(x)$  is continuous in the open  
 intervals  $(-\infty, x_1), (x_1, x_2), \dots, (x_{k-1}, x_k), (x_k, +\infty)$   
 and changes from 0 to  $-\infty$  in each of the intervals  
 $(x_{i-1}, x_i)$  and from  $+\infty$  to 0 in the interval  
 $(x_k, \infty)$ . This is clear because the roots are  
 distinct and  $\varphi(x)$  changes sign at  $x_i$ .

Therefore  $\varphi(x) + \lambda$  has a root in each of the  
 intervals  $(x_{i-1}, x_i)$  and moreover for  $\lambda > 0$  one  
 root in the interval  $(-\infty, x_1)$  and for  $\lambda < 0$  one  
 root in the interval  $(x_k, +\infty)$ .

Thus,  $\varphi(x) + \lambda$  and therefore  
 $f(x)[\varphi(x) + \lambda] = \lambda f(x) + f'(x)$  has  $k$  roots distinct from  
 $x_1, x_2, \dots, x_k$ , for  $\lambda \neq 0$ , or  $k - 1$  roots distinct  
 from  $x_1, x_2, \dots, x_k$  for  $\lambda = 0$ . Moreover  
 $\lambda f(x) + f'(x)$  has  $x_1, x_2, \dots, x_k$  as roots of  
 multiplicities  $\alpha_1 - 1, \alpha_2 - 1, \dots, \alpha_k - 1$ ,  
 respectively. This establishes the general  
 proposition that the number of real roots of the  
 polynomial  $\lambda f(x) + f'(x)$ , counted according to  
 multiplicity, is  $\alpha_1 + \alpha_2 + \dots + \alpha_k$  if  $\lambda \neq 0$ ; and  
 is  $\alpha_1 + \alpha_2 + \dots + \alpha_k - 1$  if  $\lambda = 0$ . Thus the  
 number of such roots is the same as the degree of  
 the polynomial  $\lambda f(x) + f'(x)$ .

717 Let

$$g(x) = a_0(x + \lambda_1)(x + \lambda_2) \dots (x + \lambda_n),$$

$$F_0(x) = a_0 f(x),$$

$$F_1(x) = F_0(x) + \lambda_1 F'_0(x) = a_0 f(x) + a_0 \lambda_1 f'(x),$$

$$F_2(x) = F_1(x) + \lambda_2 F'_1(x) = a_0 f(x) + a_0(\lambda_1 + \lambda_2) f'(x) + a_0 \lambda_1 \lambda_2 f''(x)$$

etc.

$$\text{Then } F_n(x) = F_{n-1}(x) + \lambda_n F'_{n-1}(x) = a_0 f(x) + a_1 f'(x) + \dots + a_n f^{(n)}(x),$$

where  $a_0, a_1, \dots, a_n$  are the coefficients of  $g$ .

By the result of problem 715, all the roots of all the polynomials  $F_0, F_1, \dots, F_n$  are real.

718 The polynomial  $a_0 x^n + a_1 m x^{n-1} + \dots + m(m-1) \dots (m-n+1) a_n$   
 $= [a_0 x^m + a_1 (x^m)' + \dots + a_n (x^m)^{(n)}] x^{n-m},$

and all the roots of  $x^m$  are real.

719 The polynomial  $a_n x^n + n a_{n-1} x^{n-1} + n(n-1) a_{n-2} x^{n-2} + \dots + a_0 n!$

has all its roots real. Therefore all the roots

$$\text{of } a_0 n! x^n + a_1 n(n-1) \dots 2 x^{n-1} + \dots + n a_{n-1} x + a_n$$

are real. Using the result of problem 718 again

we see that all the roots of the polynomial

$$a_0 n! x^n + a_1 n \cdot n(n-1) \dots 2 x^{n-1} + a_2 n(n-1) \cdot n(n-1) \dots 3 x^{n-2} + \dots + a_n n!$$

are real. It only remains to divide by  $n!$

720 All the roots of the polynomial

$$(1+x)^n = 1 + \frac{n}{1} x + \frac{n(n-1)}{1 \cdot 2} x^2 + \dots + x^n$$

are real. The problem is completed by referring to 719.

721 The polynomial  $f(x) = nx^n - x^{n-1} - x^{n-2} - \dots - 1$

has 1 as a root. The trick is to consider

$$F(x) = (x-1)f(x) = nx^{n+1} - (n+1)x^n + 1. \quad \text{Then}$$

$$F'(x) = n(n+1)(x-1)x^{n-1}. \quad \text{If } n \text{ is odd the}$$

polynomial  $F(x)$  has a single extremum, namely a minimum at  $x = 1$ . Therefore it has no other real roots except for the double root  $x = 1$ . If  $n$  is even,  $F(x)$  increases monotonically from  $-\infty$  to 1 for  $-\infty < x \leq 0$ , decreases from 1 to 0 for  $0 \leq x \leq 1$  and increases from 0 to  $\infty$  for  $1 \leq x < \infty$ . Therefore  $F(x)$  has an additional root in this case.

722 The derivative of the polynomial in this problem is positive for all real values of  $x$ . Thus the polynomial has a single real root.

723 Let  $a < b < c$ ;  $f(-\infty) < 0$ ;  $f(a) = B^2(b - a) + C^2(c - a) > 0$ ;  $f(c) = -A^2(c - a) - B^2(c - b) < 0$ ;  $f(+\infty) > 0$ . Therefore  $f$  has real roots in the intervals  $(-\infty, a)$ ;  $(a, c)$ ;  $(c, +\infty)$ .

724 Suppose

$$a_1 < a_2 < \dots < a_n.$$

Set

$$\varphi(a + bi) = B + \sum_{k=1}^n \frac{A_k^2}{a + bi - a_k} = B + \sum_{k=1}^n \frac{A_k^2(a - a_k - bi)}{(a - a_k)^2 + b^2};$$

$$\operatorname{Im}(\varphi(a + bi)) = -b \sum_{k=1}^n \frac{A_k^2}{(a - a_k)^2 + b^2} \neq 0$$

for  $b \neq 0$ , and every term in the summation sign is positive. Therefore  $\varphi(a + bi) \neq 0$  for  $b \neq 0$ .

Another way to obtain this result is to note that  $\varphi(x)$  decreases from  $+\infty$  to  $-\infty$  when  $x$  increases from  $a_1$  to  $a_{1+1}$ ;  $\varphi(x)$  decreases from 0 to  $-\infty$  on  $(-\infty, a_1)$ ;  $\varphi(x)$  decreases from  $+\infty$  to 0 on  $(a_n, \infty)$ .

725  $\frac{f'(x)}{f(x)} \sum_{k=1}^n \frac{1}{x-x_k}$  , where  $x_k$  are the roots of  $f(x)$ .

Then

$$[f'(x)]^2 - f(x)f''(x) = [f'(x)]^2 \sum_{k=1}^n \frac{1}{(x-x_k)^2} > 0$$

for every real  $x$ .

726 Let  $x_1 < x_2 < \dots < x_n$  be the roots of the polynomial  $f(x)$ , and  $y_1 < y_2 < \dots < y_m$  be roots of the polynomial  $\varphi(x)$ .

The hypothesis shows that either  $m = n$ ,  $m = n - 1$ , or  $m = n + 1$ . A simple change of notation brings us to one of the following two cases:

$$x_1 < y_1 < x_2 < y_2 < \dots < y_{n-1} < x_n$$

or

$$x_1 < y_1 < x_2 < y_2 < \dots < y_{n-1} < x_n < y_n.$$

Suppose  $\lambda \neq 0$ . We write the equation to be investigated

$$f(x)/\varphi(x) = -\mu/\lambda,$$

and set  $\psi(x) = f(x)/\varphi(x)$ .

If  $m = n$ , then  $\psi(x)$  decreases: from  $a_0/b_0$  to  $-\infty$  for  $-\infty < x < y_1$ , taking the value 0 at  $x = x_1$ ;

from  $+\infty$  to  $-\infty$  for  $y_k < x < y_{k+1}$ , taking the value 0 at  $x = x_{k+1}$ ;

from  $+\infty$  to  $a_0/b_0$  for  $y_n < x < +\infty$ .

Here we have to know that the leading coefficients of  $f(x)$ ,  $\varphi(x)$  are  $a_0$ ,  $b_0$  and assume them positive.



Now the function  $\psi(x)$  is continuous in each of the above intervals. Thus the equation  $\psi(x) = -\mu/\lambda$  has  $n$  real roots if  $-\mu/\lambda \neq a_0/b_0$ , and has  $n - 1$  real roots in the contrary case. Thus the number of real roots of the equation  $\lambda f(x) + \mu \varphi(x)$  is equal to its degree.

When  $m = n - 1$ , the argument is similar.

727 Since the polynomial  $F(x)$  reduces respectively to  $f(x)$ ,  $\varphi(x)$  in the cases  $\lambda = 1, \mu = 0$ ;  $\mu = 1, \lambda = 0$ , all roots of  $f(x)$ ,  $\varphi(x)$  are real.

Assume that the roots of  $f(x)$ ,  $\varphi(x)$  do not interlace. The only case to consider is that in which the polynomial  $f(x)$  has two successive roots  $x_1, x_2$  and the polynomial  $\varphi(x)$  has no roots in the interval  $(x_1, x_2)$ . Since the quotient

$\psi(x) = f(x)/\varphi(x)$  is continuous on the interval  $(x_1, x_2)$  in this case and reduces to zero at the end points, Rolle's theorem shows that the derivative  $\psi'(x_0)$  is 0 at an intermediate point  $x_0$  of this interval. Thus the polynomial  $\psi(x) - \psi(x_0)$  has  $x_0$  as a  $k$ -fold root,  $k \geq 2$ . The fundamental result of problem 581 shows that there is a small circle  $|z - x_0| = \rho$  on the circumference of which there are at least four points with  $\operatorname{Im}(\psi(z)) = \operatorname{Im}(\psi(x_0)) = 0$ .

Of these four points at least one,  $z_0$ , is not real. The number  $\mu = \psi(z_0)$  is real. (Just established.) The polynomial  $F(x) = -f(x) + \mu\varphi(x)$

has a nonreal root, but this contradicts the hypothesis.

- 728 Let  $\xi_1 < \xi_2 < \dots < \xi_{n-1}$  be the roots of the polynomial  $f'(x)$ . These numbers divide the real axis into  $n$  intervals:

$$(-\infty, \xi_1), (\xi_1, \xi_2), \dots, (\xi_{n-2}, \xi_{n-1}), (\xi_{n-1}, \infty).$$

By Rolle's theorem the polynomial  $f(x)$  has no more than one root in each of these intervals. Further for arbitrary  $\lambda$ , the polynomial  $f'(x) + \lambda f''(x)$  has no more than one root in each of the intervals in question. Therefore the polynomial  $f(x) + \lambda f'(x)$  has no more than two roots in each of the intervals, multiplicity being counted.

Divide the intervals into two categories. The first category consists of the intervals containing roots of  $f(x)$ . The second category consists of the intervals containing no roots of  $f(x)$ . Consider the function  $\psi(x) = f(x)/f'(x)$ . In every interval of the first category  $\psi(x)$  has a single root and therefore changes sign on the interval. In every interval of the second category  $\psi(x)$  has an invariable sign. Thus in every interval of the first category,  $\psi(x) + \lambda$  has an odd number of roots, multiplicity being counted. The only odd number no greater than 2 is 1. Therefore  $\psi(x) + \lambda$  has a single root in each interval of the first category. Therefore

$\psi'(x)$  has no root in an interval of the first category.

In an interval of the second category we reason as follows. Suppose the absolute value of  $\psi(x)$  assumes a minimum at the point  $\xi_0$  interior to an interval of the second category:  $\lambda_0 = \psi(\xi_0)$ . Suppose for definiteness that  $\psi(x)$  is positive in this interval. Then the function  $\psi(x) - \lambda$  has no root in the interval if  $\lambda < \lambda_0$ ; and at least two roots if  $\lambda > \lambda_0$ .

The arguments in the first paragraph show that for  $\lambda > \lambda_0$ ,  $\psi(x) - \lambda$  has exactly two roots in an interval of the second category and both are simple. Thus  $\psi(x) - \lambda_0$  has  $\xi_0$  as a multiple root of multiplicity two.

Summarizing, we see that  $\psi(x) - \lambda$  has no multiple roots in an interval of the second category for just one value of  $\lambda$ . Clearly every root of  $\eta f'^2(x) - f(x) f''(x)$  is a multiple root of  $\psi(x) - \psi(\eta)$ , since

$$[\psi(x)]' = \frac{f'^2(x) - f(x) f''(x)}{[f'(x)]^2}.$$

Thus the number of real roots of  $[f'(x)]^2 - f(x) f''(x)$  is the same as the number of intervals in the second category. This is clearly the number of imaginary roots of  $f(x)$ .

- 729 Problem 727 shows that the roots of  $f_1'(x)$ ,  $f_2'(x)$  interlace if all the roots of  $\lambda f_1'(x) + \mu f_2'(x)$  are real. This follows by applying Rolle's theorem to the combination  $\lambda f_1(x) + \mu f_2(x)$  since all the roots of the latter are real for arbitrary  $\lambda, \mu$ . (See problem 726.)
- 730 Let  $f(x)$  have no multiple roots, and suppose  $\xi_1 < \xi_2 < \dots < \xi_{n-1}$  are the roots of  $f'(x)$ . Consider the function  $\psi(x) = \frac{f(x)}{f'(x)} + \frac{x+\lambda}{\gamma}$ . Obviously  $\lim_{x \rightarrow \infty} \frac{\psi(x)}{x} = \frac{1}{n} + \frac{1}{\gamma} > 0$ , for  $\gamma > 0$ , and for  $\gamma < -n$ . Thus  $\psi(x) \rightarrow +\infty$  for  $x \rightarrow +\infty$ , and  $\psi(x) \rightarrow -\infty$  for  $x \rightarrow -\infty$ . Moreover,  $\psi(x) \rightarrow -\infty$  for  $x \rightarrow \xi_i +$ ;  $\psi(x) \rightarrow +\infty$  for  $x \rightarrow \xi_i -$ . Thus  $\psi(x)$  increases from  $-\infty$  to  $+\infty$  in each of the intervals  $(-\infty, \xi_1)$ ,  $(\xi_1, \xi_2)$ ,  $\dots$ ,  $(\xi_{n-1}, \infty)$ , and is continuous in the interiors of these intervals.

Thus  $\psi(x)$ , and therefore its numerator  $\gamma f(x) + (x+\lambda)f'(x)$  have no fewer than  $n$  distinct roots for  $\gamma > 0$ , or for  $\gamma < -n$ . Now  $\gamma f(x) + (x+\lambda)f'(x)$  is a polynomial of degree  $n$ ; thus it has at most  $n$  roots. If  $f(x)$  has a multiple root and  $x_1, x_2, \dots, x_k$  are the distinct roots of  $f(x)$ , then  $f'(x)$  has  $k-1$  roots  $\xi_1, \xi_2, \dots, \xi_{k-1}$ , distinct from  $x_1, x_2, \dots, x_k$ . This is a general fact that follows from the study of the graph of the polynomial. Now we simply

factor off the necessary factors  $(x - x_1)^j, \dots$ , from  $\phi(x)$  and use the arguments above for the complementary factors. To do this we have only to know that  $\xi_1, \xi_2, \dots, \xi_{k-1}$  are not roots of  $f(x)$  unless  $-\lambda$  is a root of  $f(x)$ . If  $-\lambda$  is not a root of  $f(x)$ , the number of real roots of  $\gamma f(x) + (x + \lambda)f'(x)$  is  $n - k$  (to allow for the multiple roots) plus  $k$ , to allow for the remaining roots. If  $-\lambda$  is a root of  $f(x)$ , the first summand becomes  $n - k + 1$ , the second summand  $k - 1$ .

Thus the polynomial  $\gamma f(x) + (\lambda + x)f'(x)$  has  $n$  roots in all cases, multiplicity being counted.

731 Set  $\varphi(x) = b_k(x + \gamma_1)(x + \gamma_2) \dots (x + \gamma_k)$ .

Each of the numbers  $\gamma_i$  is either positive or less than  $-n$ :  $\gamma_i > 0$  or  $\gamma_i < -n$ .

Obviously the coefficients of the polynomial

$$F_1(x) = \gamma_1 f(x) + x f'(x) \text{ are } a_i(\gamma_1 + i),$$

and the coefficients of the polynomial

$$F_2(x) = \gamma_2 F_1(x) + x F_1'(x) \text{ are } a_i(\gamma_1 + i)(\gamma_2 + i);$$

etc. The coefficients of the polynomial

$$F_k(x) = \gamma_k F_{k-1}(x) + x F_{k-1}'(x)$$

are

$$a_i(\gamma_1 + i)(\gamma_2 + i) \dots (\gamma_k + i), i = 1, 2, \dots, n.$$

The fundamental result of problem 730 shows that all roots of all the polynomials  $F_1, F_2, \dots, F_k$

are real. But

$$a_0\varphi(0) + a_1\varphi(1)x + \dots + a_n\varphi(n)x^n = b_k F_k(x),$$

- 732 Set  $f(x) = f_1(x)(x + \lambda)$ , where  $\lambda$  is real and  $f_1(x)$  is a polynomial of degree  $(n - 1)$ , having real roots. As an induction hypothesis suppose the theorem established for polynomials of degree  $(n - 1)$ .

Set

$$\begin{aligned} f_1(x) &= b_0 + b_1x + \dots + b_{n-1}x^{n-1}, \\ f(x) &= a_0 + a_1x + \dots + a_nx^n. \end{aligned}$$

Then

$$\begin{aligned} a_0 &= \lambda b_0, \\ a_1 &= \lambda b_1 + b_0, \\ a_2 &= \lambda b_2 + b_1, \\ &\dots \dots \dots \\ a_{n-1} &= \lambda b_{n-1} + b_{n-2}, \\ a_n &= b_{n-1} \end{aligned}$$

and

$$\begin{aligned} &a_0 + a_1\gamma x + a_2\gamma(\gamma - 1)x^2 + \dots + a_n\gamma(\gamma - 1)\dots(\gamma - n + 1)x^n \\ &= \lambda[b_0 + b_1\gamma x + b_2\gamma(\gamma - 1)x^2 + \dots + b_{n-1}\gamma(\gamma - 1)\dots \\ &\dots(\gamma - n + 2)x^{n-1}] + x\gamma(b_0 + b_1(\gamma - 1)x + b_2(\gamma - 1)(\gamma - 2)x^2 + \dots \\ &\dots + b_{n-1}(\gamma - 1)(\gamma - 2)\dots(\gamma - n + 1)x^{n-1}) \\ &= \lambda\varphi(x) + x[\gamma\varphi(x) - x\varphi'(x)], \end{aligned}$$

where  $\varphi(x)$  denotes the polynomial

$$b_0 + b_1\gamma x + b_2\gamma(\gamma - 1)x^2 + \dots + b_{n-1}\gamma(\gamma - 1)\dots(\gamma - n + 2)x^{n-1}.$$

The induction hypothesis allows us to assume that all roots of the polynomial  $\varphi(x)$  are real. To complete the argument we shall establish the following lemma.

Lemma. If  $\varphi(x)$  is a polynomial of degree  $n - 1$  having real roots, then all roots of the polynomial  $\psi(x) = \lambda\varphi + \gamma x\varphi - x^2\varphi'$  are real for  $\gamma > n - 1$ , for arbitrary real  $\lambda$ .

Proof. If 0 is a root of  $\varphi$ , we can factor it off as follows:  $\varphi = x^k\varphi_1$ ,  $\varphi_1(0) \neq 0$ ,

$$\psi(x) = x^k(\lambda\varphi_1 + (\gamma - k)x\varphi_1 - x^2\varphi_1') = x^k\psi_1,$$

and  $\gamma_1 = \gamma - k$  exceeds the degree of  $\varphi_1$ .

Now suppose that 0 is not a root of  $\varphi$ , and let  $x_1, x_2, \dots, x_m$  be its distinct roots. The polynomial  $\psi$  will have these same numbers as roots, with multiplicity sum  $n - 1 - m$ . Now consider

$$w(x) = \lambda + \gamma x - \frac{x^2\varphi'(x)}{\varphi(x)}.$$

It is obvious that

$$\lim_{x \rightarrow \infty} \frac{w(x)}{x} = \gamma - (n - 1) > 0.$$

Therefore  $w(x) \rightarrow -\infty$  for  $x \rightarrow -\infty$  and  $w(x) \rightarrow +\infty$  for  $x \rightarrow +\infty$ . Moreover  $w(x) \rightarrow +\infty$  for  $x \rightarrow x_i +$ ,  $w(x) \rightarrow -\infty$  for  $x \rightarrow x_i -$ . Thus  $w(x)$  has roots in each of the intervals

$$(-\infty, x_1), (x_1, x_2), \dots, (x_{m-1}, x_m), (x_m, +\infty).$$

It follows that the number of real roots of  $\psi(x)$ , multiplicity being counted is  $n - 1 - m + m + 1 = n$ . This is the degree of  $\psi(x)$ , as was to be shown.

- 733 If all roots of the polynomial  $a_0 + a_1x + \dots + a_nx^n$  are real, then all roots of the polynomial  $a_0x^n + a_1x^{n-1} + \dots + a_n$  are also real. Now all roots of the polynomials

$$a_0\gamma_1(\gamma_1-1)\dots(\gamma_1-n+1)x^n + a_1\gamma_1(\gamma_1-1)\dots(\gamma_1-n+2)x^{n-1} + \dots + a_{n-1}\gamma_1x + a_n$$

and

$$\begin{aligned} & a_0\gamma_1(\gamma_1-1)\dots(\gamma_1-n+1) + a_1\gamma_1(\gamma_1-1)\dots(\gamma_1-n+2)x + \dots \\ & \dots + a_{n-1}\gamma_1x^{n-1} + a_nx^n = \left[ a_0 + \frac{a_1}{\gamma_1-n+1}x + \dots \right. \\ & \quad \left. + \frac{a_{n-1}}{(\gamma_1-n+1)(\gamma_1-n+2)\dots(\gamma_1-1)}x^{n-1} \right. \\ & \quad \left. + \frac{a_n}{(\gamma_1-n+1)(\gamma_1-n+2)\dots\gamma_1}x^n \right] \gamma_1(\gamma_1+1)\dots(\gamma_1-n+1) \end{aligned}$$

are real for  $\gamma_1 > n-1$ . Set  $\gamma_1 - n + 1 = \alpha > 0$ ,

and note that all roots of the polynomial

$$a_0 + \frac{a_1}{\alpha}x + \frac{a_2}{\alpha(\alpha+1)}x^2 + \dots + \frac{a_n}{\alpha(\alpha+1)\dots(\alpha+n-1)}x^n$$

are real. The result of the exercise follows from a second application of problem 732.

- 734 1. Suppose that all roots of  $f(x)$  are positive. Then the polynomial  $a_0 + a_1wx + \dots + a_nw^{n^2}x^n$  has no negative roots.

Suppose the theorem established for polynomials of degree  $n-1$ . Suppose

$$\varphi(x) = b_0 + b_1wx + b_2w^4x^2 + \dots + b_{n-1}w^{(n-1)^2}x^{n-1}.$$

Set  $0 < x_1 < x_2 < \dots < x_{n-1}$ , where  $x_1, x_2, \dots, x_{n-1}$  are the roots of  $\varphi(x)$  and suppose  $\frac{x_i}{x_{i-1}} > w^{-2}$ .



Now set  $f(x) = (\lambda - x)(b_0 + b_1x + \dots + b_{n-1}x^{n-1})$ .

The coefficients of the polynomial  $f(x)$  are

$$\begin{aligned} a_0 &= \lambda b_0, \\ a_1 &= \lambda b_1 - b_0, \\ a_2 &= \lambda b_2 - b_1, \\ &\dots \dots \dots \\ a_{n-1} &= \lambda b_{n-1} - b_{n-2}, \\ a_n &= -b_{n-1}. \end{aligned}$$

Thus

$$\begin{aligned} \psi(x) &= a_0 + a_1wx + a_2w^2x^2 + \dots + a_nw^{n-1}x^n = \lambda(b_0 + b_1wx + \dots \\ &\dots + b_{n-1}w^{n-1}x^{n-1}) - x(b_0w + b_1w^2x + \dots + b_{n-1}w^{n-1}x^{n-1}) \\ &= \lambda\varphi(x) - xw\varphi(xw^2). \end{aligned}$$

The roots of the polynomial  $\varphi(x)$ ,  $xw\varphi(xw^2)$  interlace by the induction hypothesis. Therefore all the roots of the polynomial  $\lambda\varphi(x) + xw\varphi(xw^2)$  are real. It remains to check that they are distributed in the same way as the roots of the polynomial  $\varphi(x)$ .

Suppose the roots of  $\psi(x)$  are  $z_1, z_2, \dots, z_n$ .

It is easy to see that

$$\begin{aligned} 0 < z_1 < x_1 < x_1w^{-2} < z_2 < x_2 < x_2w^{-2} < z_3 \\ &\dots < z_{n-1} < x_{n-1} < x_{n-1}w^{-2} < z_n. \end{aligned}$$

Furthermore,  $\frac{z_l}{z_{l-1}} > \frac{x_{l-1}w^{-2}}{x_{l-1}} = w^{-2}$ , and the assertion is established.

2. Consider

$$\varphi_m(x) = \left(1 - \frac{x^2 \lg \frac{1}{w}}{m}\right)^m.$$

If  $m$  is sufficiently large, the roots of the polynomial  $\varphi_m(x)$  are  $\pm \sqrt{\frac{m}{\lg \frac{1}{w}}}$ ; these do

not lie in the interval  $(0, n)$ . Thus, by problem 731, all the roots of the polynomial  $a_0\varphi_m(0) + a_1\varphi_m(1)x + \dots + a_n\varphi_m(n)x^n$  are real. But  $\lim_{m \rightarrow \infty} \varphi_m(x) = w^{x^2}$ . Thus all the roots of  $a_0 + a_1wx + \dots + a_nw^{n^2}x^n$  are real, since the roots of a polynomial are continuous functions of the coefficients.

735 Let the roots of the polynomial

$$f(x) = a_0 + a_1x + \dots + a_nx^n \quad \text{be } x_1, x_2, \dots, x_n.$$

Without loss of generality we may assume these to be positive. Now set

$$\varphi(x) = a_0 \cos \varphi + a_1 \cos(\varphi + \theta)x + \dots + a_n \cos(\varphi + n\theta)x^n,$$

$$\psi(x) = a_0 \sin \varphi + a_1 \sin(\varphi + \theta)x + \dots + a_n \sin(\varphi + n\theta)x^n.$$

Then

$$\varphi(x) + i\psi(x) = (\cos \varphi + i \sin \varphi) a_n \prod_{i=1}^n (\alpha x - x_i),$$

$$\varphi(x) - i\psi(x) = (\cos \varphi - i \sin \varphi) a_n \prod_{i=1}^n (\alpha' x - x_i),$$

where

$$\alpha = \cos \theta + i \sin \theta, \quad \alpha' = \cos \theta - i \sin \theta.$$

Thus

$$\Phi(x) = \frac{\varphi(x) + i\psi(x)}{\varphi(x) - i\psi(x)} = \frac{\cos \varphi + i \sin \varphi}{\cos \varphi - i \sin \varphi} \prod_{i=1}^n \frac{\alpha x - x_i}{\alpha' x - x_i}.$$

Let  $x = \rho\beta$  be a root of the polynomial  $\varphi(x)$ ;

$$\rho = |x|; \quad \beta = \cos \lambda + i \sin \lambda. \quad \text{Then } |\Phi(x)| = 1,$$

and therefore

$$\prod_{i=1}^n \left| \frac{\rho\alpha\beta - x_i}{\rho\alpha'\beta - x_i} \right| = 1,$$

But

$$\begin{aligned} \left| \frac{\rho\alpha\beta - x_i}{\rho\alpha'\beta - x_i} \right|^2 &= \frac{(\rho\alpha\beta - x_i)(\rho\alpha'\beta' - x_i)}{(\rho\alpha'\beta - x_i)(\rho\alpha\beta' - x_i)} = \\ &= 1 + \frac{\rho x_i (\alpha - \alpha')(\beta' - \beta)}{|\rho\alpha'\beta - x_i|^2} = 1 + \frac{4\rho x_i \sin \theta \sin \lambda}{|\rho\alpha'\beta - x_i|^2}. \end{aligned}$$

The case  $\sin \theta = 0$  is not interesting.

If  $\sin \lambda \neq 0$ , then all the expressions

$\left| \frac{\rho\alpha\beta - x_i}{\rho\alpha'\beta - x_i} \right|^2$  are simultaneously greater than unity or simultaneously less than unity and their product cannot be 1. Therefore we must have  $\sin \lambda = 0$ , that is,  $x$  real.

- 736 Let  $x_1, x_2, \dots, x_n$  be roots of the polynomial  $f(x) = a_0 + ib_0 + i(a_1 + ib_1)x + \dots + (a_n + ib_n)x^n = \varphi(x) + i\psi(x)$ . Each of these roots has positive imaginary part. Consider the polynomial  $\bar{f}(x) = \varphi(x) - i\psi(x)$ . Clearly this polynomial has roots  $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$ .

Thus

$$\Phi(x) = \frac{\varphi(x) + i\psi(x)}{\varphi(x) - i\psi(x)} = \prod_{i=1}^n \frac{x - x_i}{x - \bar{x}_i} \cdot \frac{a_n + ib_n}{a_n - ib_n}.$$

For every root  $x_0$  of the polynomial  $\varphi(x)$  the relation

$$|\Phi(x_0)| = \prod_{i=1}^n \left| \frac{x_0 - x_i}{x_0 - \bar{x}_i} \right| = 1 \text{ holds.}$$

But

$$\begin{aligned} \left| \frac{x_0 - x_l}{x_0 - x'_l} \right|^2 &= \frac{(x_0 - x_l)(x'_0 - x'_l)}{(x_0 - x'_l)(x'_0 - x_l)} = 1 + \frac{(x_l - x'_l)(x_0 - x'_0)}{|x_0 - x_l|^2} \\ &= 1 - \frac{4 \operatorname{Im}(x_0) \operatorname{Im}(x_l)}{|x_0 - x'_l|^2}. \end{aligned}$$

A simple geometric argument shows that if

$\operatorname{Im}(x_0) > 0$ , then  $\left[ \frac{x_0 - x_1}{x_0 - \bar{x}_1} \right] < 1$ ; in the contrary

case this fraction is always greater than 1.

Therefore the equality  $|\Phi(x_0)| = 1$  can be valid only if  $x_0$  is real. That is, all roots of  $\varphi(x)$  are real.

Now consider the polynomial

$$(\alpha - \beta i)[\varphi(x) + i\psi(x)] = \alpha\varphi(x) + \beta\psi(x) + i[\alpha\psi(x) - \beta\varphi(x)].$$

Its roots are the same as the roots of the polynomial under consideration. Therefore the real part,  $\alpha\varphi(x) + \beta\psi(x)$  has real roots for arbitrary real  $\alpha, \beta$  (not both zero). By problem 727 the roots of  $\varphi(x), \psi(x)$  interlace.

- 737 Let  $\varphi(x)$  have roots  $x_1, x_2, \dots, x_n$ ; and let  $\psi(x)$  have roots  $y_1, y_2, \dots, y_n$ . It can be assumed without loss of generality that the leading coefficients are positive, and that

$$x_1 > y_1 > x_2 > y_2 > \dots > y_{n-1} > x_n > y_n$$

where the last member may be absent.

We expand  $\psi(x)/\varphi(x)$  into partial fractions:

$$\frac{\psi(x)}{\varphi(x)} = A + \sum_{k=1}^n \frac{A_k}{x - x_k}; \quad A_k = \frac{\psi(x_k)}{\varphi'(x_k)}.$$

It is easy to see that every numerator  $A_k > 0$ .

Take  $x = a + bi$  and compute the imaginary part of

$$\frac{-i(\varphi(x) + i\psi(x))}{\varphi(x)} = \frac{\psi(x)}{\varphi(x)} - i;$$

$$\operatorname{Im}\left(\frac{\psi(x)}{\varphi(x)} - i\right) = -1 + \operatorname{Im}\left(\sum_{k=1}^n \frac{A_k}{a + bi - x_k}\right) = -1 - b \sum_{k=1}^n \frac{A_k}{(a - x_k)^2 + b^2}.$$

If  $b \geq 0$ , then  $\operatorname{Im}\left(\frac{\psi(x)}{\varphi(x)} - i\right) < 0$ ; therefore

$\varphi(x) + i\psi(x) \neq 0$ . Thus all roots of the polynomial

$\varphi(x) + i\psi(x)$  would lie in the lower half plane.

Similar arguments take care of the remaining cases.

738

$$\frac{f'(x)}{f(x)} = \sum_{k=1}^n \frac{1}{x - x_k}; \text{ where } x_k \text{ are the roots of } f(x).$$

Set  $x = a - bi$ ,  $b > 0$ . Then

$$\operatorname{Im}\left(\frac{f'(a - bi)}{f(a - bi)}\right) = \sum_{k=1}^n \frac{b + \operatorname{Im}(x_k)}{(x - x_k)^2} > 0.$$

Thus

$$f'(a - bi) \neq 0.$$

739 Let the given half plane be defined by the inequality

$$r \cos(\theta - \varphi) > p, \text{ where } x = r(\cos \varphi + i \sin \varphi).$$

Set  $x = (\bar{x} + \pi i)(\sin \theta - i \cos \theta)$ . Then

$$\begin{aligned} \bar{x} &= -\pi i + x (\sin \theta + i \cos \theta = r \sin(\theta - \varphi) \\ &\quad + i[r \cos(\theta - \varphi) - p]). \end{aligned}$$

It is clear that if  $x$  lies in the given half plane, then  $\bar{x}$  lies in the half plane  $\operatorname{Im}(\bar{x}) > 0$ ,

and conversely. The roots of the polynomials  $f[(\bar{x} + \pi)(\sin \theta - i \cos \theta)]$ , must therefore lie in the upper half plane. Problem 738 shows that the roots of its derivative, namely the roots of  $[\sin \theta - i \cos \theta] f'[(\bar{x} + \pi)(\sin \theta - i \cos \theta)]$ , must also lie in the upper half plane.

Therefore the roots of the polynomial  $f'(x)$  lie in the given half plane.

740 Follows directly from problem 739.

741 The roots of the given polynomial satisfy one of the equations:

$$\frac{f'(x)}{f(x)} + \frac{1}{ki} = 0 \quad , \quad \frac{f'(x)}{f(x)} - \frac{1}{ki} = 0.$$

If we separate these into partial fractions we obtain the two equations  $\sum_{k=1}^n \frac{1}{x-x_k} \pm \frac{1}{ki} = 0$ ,

where  $x_k$  are the roots of  $f(x)$ , which are real by hypothesis. Set  $x = a + bi$ . Then

$$\left| \operatorname{Im} \left( \sum_{k=1}^n \frac{1}{x-x_k} \right) \right| = |b| \sum_{k=1}^n \frac{1}{(a-x_k)^2 + b^2} < \frac{n}{|b|}.$$

Thus every root of the given polynomial must satisfy  $1/k < n/|b|$ . Thus  $|b| < kn$ .

742 Denote the roots of  $f'(x)$  by  $\xi_1, \xi_2, \dots, \xi_{n-1}$ ; these are obviously real. Let the roots of  $f(x) - b, f(x) - a$  be  $y_1, y_2, \dots, y_n; x_1, x_2, \dots, x_n$  respectively. Then

$$y_1 < \xi_1 < y_2 < \xi_2 < \dots < y_{n-1} < \xi_{n-1} < y_n, \\ x_1 < \xi_1 < x_2 < \xi_2 < \dots < x_{n-1} < \xi_{n-1} < x_n.$$

It follows from these equations that the intervals bounded by the points  $x_i, y_i$  do not overlap. Thus they are all included in one or another of the non-overlapping intervals

$$(-\infty, \xi_1); (\xi_1, \xi_2); \dots; (\xi_{n-1}, +\infty).$$

In each interval  $x_i, y_i$  the polynomial  $f(x)$  starts at the value  $a$  and changes continuously to the value  $b$ . Therefore the polynomial  $f(x) - \lambda$  is 0 for  $n$  distinct real values. The assertion is established if all roots are distinct. A very simple modification of the argument covers the case of multiple roots.

- 743 If the real parts of the roots of the polynomial  $f(x) = x^n + a_1x^{n-1} + \dots + a_n$  have the same sign, the imaginary parts of the roots of the polynomial

$$i^n f(-ix) = x^n + ia_1x^{n-1} - a_2x^{n-2} - ia_3x^{n-3} + \dots$$

also have the same sign. The converse is true.

By problems 736, 737 necessary and sufficient conditions for this are that the roots of the polynomials  $x^n - a_2x^{n-2} + a_4x^{n-4} - \dots$ ,  $a_1x^{n-1} - a_3x^{n-3} + a_5x^{n-5} - \dots$  be real and interlace.

- 744 It must be true that  $a > 0$  and that the roots of the polynomials  $x^3 - bx, ax^2 - c$  be real and interlace. Necessary and sufficient conditions for this are  $0 < c/a < b$ , or  $c > 0, ab - c > 0$ .

Thus the conditions  $a > 0$ ,  $c > 0$ ,  $ab - c > 0$  are necessary and sufficient conditions that the real parts of the roots of the given equation be negative.

745  $a > 0$ ,  $c > 0$ ,  $d > 0$ ,  $abc - c^2 - a^2d > 0$ .

746 Set  $x = (1 + y)/(1 - y)$ . It is easy to see that if  $|x| < 1$ , then the real part of  $y$  is negative and conversely.

Thus all three roots,  $x_1, x_2, x_3$  of the equation  $f(x) = 0$  will have absolute value less than 1, if and only if all roots of the equation  $f(\{1 + y\}/\{1 - y\}) = 0$  have negative real part.

This equation has the form

$$y^3(1 - a + b - c) + y^2(3 - a - b + 3c) + y(3 + a - b - 3c) + (1 + a + b + c) = 0.$$

It is easy to see moreover that the condition

$$1 - a + b - c = (1 + x_1)(1 + x_2)(1 + x_3) > 0$$

is necessary. Using the result of problem 744 we obtain the following necessary and sufficient conditions

$$1 - a + b - c > 0; \quad 1 + a + b + c > 0; \quad 3 - a - b + 3c > 0; \\ 1 - b + ac - c^2 > 0.$$

747  $f(x)(1 - x) = a_n + (a_{n-1} - a_n)x + \\ + (a_{n-2} - a_{n-1})x^2 + \dots + (a_0 - a_1)x^n - a_0x^{n+1}.$

Set  $|x| = \rho > 1$ . Then

$$|f(x)(1 - x)| \geq a_0\rho^{n+1} - |a_n + (a_{n-1} - a_n)x + \dots + \\ + (a_0 - a_1)x^n| \geq a_0\rho^{n+1} - \rho^n(a_n + a_{n-1} - a_n + \dots + \\ + a_0 - a_1) = a_0(\rho^{n+1} - \rho^n) > 0.$$



Thus,  $f(x) \neq 0$  for  $|x| > 1$ .

748 - 0.6618.

749 2.094551.

750 a) 3.3876, -0.5136, -2.8741;

b) 2.8931;

c) 3.9489, 0.2172, -1.1660;

d) 3.1149, 0.7459, -0.8608.

751 The problem comes down to calculating the root of the equation  $x^3 - 3x + 1 = 0$  lying the interval  $(0, 1)$ .

Answer:  $x = 0.347$ , to within 0.001.

752 2.4908.

753 a) 1.7320; b) -0.7321; c) 0.6180; d) 0.2679;

e) -3.1623; f) 1.2361; g) -2.3028; h) 3.6457;

i) 1.6180.

754 a) 1.0953, -0.2624, -1.4773, -2.3556;

b) 0.8270, 0.3383, -1.2090, -2.9563;

c) 1.4689, 0.1168;

d) 8.0060, 1.2855, 0.1960, -1.4875;

e) 1.5357, -0.1537;

f) 3.3322, 1.0947, -0.6002, -1.8268;

g) 0.4910, -1.4910;

h) 2.1462, -0.6821, -1.3178, -4.1463.

CHAPTER VI - SOLUTIONS  
SYMMETRIC FUNCTIONS

755 We write out the solution of problem f) in detail:

$$F(x_1, x_2, x_3) = (x_1^2 + x_2^2)(x_1^2 + x_3^2)(x_2^2 + x_3^2).$$

The highest term in the polynomial  $F$  is  $x_1^4 \cdot x_2^2$ .

The homogeneous sixth degree polynomial  $F$  can be expressed as the sum of homogeneous sixth degree polynomials made up from

$$f_1 = x_1 + x_2 + x_3$$

$$f_2 = x_1x_2 + x_2x_3 + x_3x_1$$

$$f_3 = x_1x_2x_3 \text{ in the following combinations:}$$

$$f_1f_2f_3, f_1^2f_2^2, f_1^3f_3, f_2^3, f_3^2. \text{ Thus } F \text{ can be written:}$$

$F = f_1^2f_2^2 + Af_1^3f_3 + Bf_2^3 + Cf_1f_2f_3 + Df_3^2$ , where  $A, B, C, D$  are numerical coefficients. To determine the numerical coefficients we give particular values to  $x_1, x_2, x_3$ :

$x_1$	$x_2$	$x_3$	$f_1$	$f_2$	$f_3$	$F$
1	1	0	2	1	0	2
2	-1	-1	0	-3	2	50
1	-2	-2	-3	0	4	200
1	-1	-1	-1	-1	1	8

Thus  $A, B, C, D$  must satisfy the system:

$$\begin{aligned} 2 &= 4 + B, \\ 50 &= -27B + 4D, \\ 200 &= -108A + 16D, \\ 8 &= 1 - A - B + C + D. \end{aligned}$$

and so  $B = -2$ ,  $D = -1$ ,  $A = -2$ ,  $C = 4$ .

The result is

$$(x_1^2 + x_2^2)(x_1^2 + x_3^2)(x_2^2 + x_3^2) = f_1^2 f_2^2 - 2f_1^3 f_3 - 2f_2^3 + 4f_1 f_2 f_3 - f_3^2.$$

For the remaining problems we give only the answers:

- a)  $f_1^3 - 3f_1 f_2$ ; b)  $f_1 f_2 - 3f_3$ ; c)  $f_1^4 - 4f_1^2 f_2 + 8f_1 f_3$ ;  
 d)  $f_1^3 f_2^2 - 2f_1^4 f_3 - 3f_1 f_2^3 + 6f_1^2 f_2 f_3 + 3f_2^2 f_3 - 7f_1 f_3^2$ ;  
 e)  $f_1 f_2 - f_3$ ; g)  $2f_1^3 - 9f_1 f_2 + 27f_3$ ;  
 h)  $f_1^2 f_2^2 - 4f_1^3 f_3 - 4f_2^3 + 18f_1 f_2 f_3 - 27f_3^2$ .

- 756 a)  $f_1 f_2 f_3 - f_1^2 f_4 - f_3^2$ ; b)  $f_1^2 f_4 + f_3^2 - 4f_2 f_4$ ;  
 c)  $f_1^3 - 4f_1 f_2 + 8f_3$ .

- 757 a)  $f_1^2 - 2f_2$ ; b)  $f_1^3 - 3f_1 f_2 + 3f_3$ ;  
 c)  $f_1 f_3 - 4f_4$ ; d)  $f_2^2 - 2f_1 f_3 + 2f_4$ ;  
 e)  $f_1^2 f_2 - f_1 f_3 - 2f_2^2 + 4f_4$ ; f)  $f_1^4 - 4f_1^2 f_2 + 2f_2^2 + 4f_1 f_3 - 4f_4$ ;  
 g)  $f_2 f_3 - 3f_1 f_4 + 5f_5$ ; h)  $f_1^2 f_3 - 2f_2 f_3 - f_1 f_4 + 5f_5$ ;  
 i)  $f_1 f_2^2 - 2f_1^2 f_3 - f_2 f_3 + 5f_1 f_4 - 5f_5$ ;  
 j)  $f_1^3 f_2 - 3f_1 f_2^2 - f_1^2 f_3 + 5f_2 f_3 + f_1 f_4 - 5f_5$ ;  
 k)  $f_1^5 - 5f_1^3 f_2 + 5f_1 f_2^2 + 5f_1^2 f_3 - 5f_2 f_3 - 5f_1 f_4 + 5f_5$ ;  
 l)  $f_2 f_4 - 4f_1 f_5 + 9f_6$ ; m)  $f_3^2 - 2f_2 f_4 + 2f_1 f_5 - 2f_6$ ;  
 n)  $f_1^2 f_4 - 2f_2 f_4 - f_1 f_5 + 6f_6$ ;  
 o)  $f_1 f_2 f_3 - 3f_1^2 f_4 - 3f_3^2 + 4f_2 f_4 + 7f_1 f_5 - 12f_6$ ;  
 p)  $f_2^3 - 3f_1 f_2 f_3 + 3f_1^2 f_4 + 3f_3^2 - 3f_2 f_4 - 3f_1 f_5 + 3f_6$ ;  
 q)  $f_1^3 f_3 - 3f_1 f_2 f_3 - f_1^2 f_4 + 3f_3^2 + 2f_2 f_4 + f_1 f_5 - 6f_6$ ;  
 r)  $f_1^2 f_2^2 - 2f_1^3 f_3 - 2f_2^3 + 4f_1 f_2 f_3 + 2f_1^2 f_4 - 3f_3^2$   
 $+ 2f_2 f_4 - 6f_1 f_5 + 6f_6$ ;  
 s)  $f_1^4 f_2 - 4f_1^2 f_2^2 - f_1^3 f_3 + 2f_2^3 + 7f_1 f_2 f_3$   
 $+ f_1^2 f_4 - 3f_3^2 - 6f_2 f_4 - f_1 f_5 + 6f_6$ ;  
 t)  $f_1^6 - 6f_1^4 f_2 + 9f_1^2 f_2^2 + 6f_1^3 f_3 - 2f_2^3 - 12f_1 f_2 f_3 - 6f_1^2 f_4$   
 $+ 3f_3^2 + 6f_2 f_4 + 6f_1 f_5 - 6f_6$ .

$$758 \quad \text{a) } nf_1^2 - 8f_2; \quad \text{b) } -f_1^n + 4f_1^{n-2}f_2 - 8f_1^{n-3}f_3 + \dots + (-2)^n f_n.$$

$$759 \quad \text{a) } (n-1)f_1^2 - 2nf_2; \quad \text{b) } (n-1)f_1^3 - 3(n-2)f_1f_2 + 3(n-4)f_3; \\ \text{c) } (n-1)f_1^4 - 4nf_1^2f_2 + 2(n+6)f_2^2 + 4(n-3)f_1f_3 - 4nf_4; \\ \text{d) } \frac{3(n-1)(n-2)}{2}f_1^2 - (3n-1)(n-2)f_2.$$

$$760 \quad f_k^2 - 2f_{k-1}f_{k+1} + 2f_{k-2}f_{k+2} - 2f_{k-3}f_{k+3} + \dots$$

$$761 \quad (n-1)! \sum_{i=1}^n a_i^2 f_1^2 - 2(n-2)! \left[ n \sum_{i=1}^n a_i^2 - \left( \sum_{i=1}^n a_i \right)^2 \right] f_2 \\ = (n-1)! S_2 s_2 + 4(n-2)! F_2 f_2,$$

$$\text{where } S_2 = \sum_{i=1}^n a_i^2; \quad s_2 = \sum_{i=1}^n x_i^2; \quad F_2 = \sum_{i < k} a_i a_k; \quad f_2 = \sum_{i < k} x_i x_k.$$

$$762 \quad \text{a) } \frac{f_1 f_2 - 3f_3}{f_3}; \quad \text{b) } \frac{2(f_1^2 f_2 - 3f_1 f_3 - 2f_2^2)}{f_1 f_2 - f_3}; \quad \text{c) } \frac{f_2^3 + f_1^3 f_3 - 6f_1 f_2 f_3 + 9f_3^2}{f_3^2}.$$

$$763 \quad \text{a) } \frac{f_2^2 - 2f_1 f_3 + 2f_4}{f_4}; \quad \text{b) } \frac{f_1^2 f_2^2 + f_1^3 f_3 - 6f_1 f_2 f_3 + 6f_3^2 + 2f_1^2 f_4}{f_1 f_2 f_3 - f_1^2 f_4 - f_3^2}.$$

$$764 \quad \text{a) } \frac{f_{n-1}}{f_n}; \quad \text{b) } \frac{f_{n-1}^2 - 2f_{n-2}f_n}{f_n^2}; \quad \text{c) } \frac{f_1 f_{n-1} - n f_n}{f_n}; \\ \text{d) } \frac{f_1^2 f_{n-1}^2 - 2f_2 f_{n-1}^2 - 2f_1^2 f_{n-2} f_n + 4f_2 f_{n-2} f_n - n f_n^2}{f_n^2}; \\ \text{e) } \frac{f_1^2 f_{n-1} - 2f_2 f_{n-1} - f_1 f_n}{f_n}; \quad \text{f) } \frac{f_2 f_{n-1} - (n-1)f_1 f_n}{f_n}.$$

$$765 \quad -4.$$

$$766 \quad -35.$$

767 16.

768 a)  $-3$ ; b)  $-2p^3 - 3q^2$ ; c)  $-p^3 (x_1^2 - x_2x_3 = -p)$ ;  
 d)  $q^4$ ; e)  $\frac{-2p-3q}{1+p-q}$ ; f)  $\frac{2p^2-4p-4pq+3q^2+6q}{(1+p-q)^2}$ .

769 Let  $x_1^2 = x_2^2 + x_3^2$ . Then  $2x_1^2 = x_1^2 + x_2^2 + x_3^2 = a^2 - 2b$ .

Thus either  $\sqrt{\frac{a^2-2b}{2}}$  or  $-\sqrt{\frac{a^2-2b}{2}}$  is a

root of the given equation. Necessary and sufficient conditions that this occur are:

$$a^4(a^2 - 2b) = 2(a^3 - 2ab + 2c)^2.$$

770

$$\begin{aligned} a &= -x_1 - x_2 - x_3, \\ ab - c &= -(x_1 + x_2)(x_1 + x_3)(x_2 + x_3), \\ c &= -x_1x_2x_3. \end{aligned}$$

If all the roots are real and negative then

$$a > 0, b > 0, c > 0.$$

If one root  $x_1$  is real, and  $x_2, x_3$  are complex conjugates with negative real part, then

$$x_2 + x_3 < 0, x_2x_3 > 0, (x_1 + x_2)(x_1 + x_3) > 0.$$

Thus the conditions  $a > 0, b > 0, c > 0$  again hold.

Now suppose that  $a > 0, b > 0, c > 0$ . If  $x_1$  is real and  $x_2, x_3$  are complex conjugates, then  $x_2x_3 > 0, (x_1 + x_2)(x_1 + x_3) > 0$ . Thus, since  $c > 0, b > 0$ , it follows that  $x_1 < 0$ ,  
 $2 \operatorname{Re}(x_2) = x_2 + x_3 < 0$ .

On the other hand if all of  $x_1, x_2, x_3$  are real, then it follows, since  $c > 0$ , that one of the roots  $x_1$  is negative, and the other two have the same sign. If  $x_2 > 0, x_3 > 0$ , then

$$-x_1 - x_2 > x_3 > 0, \quad -x_1 - x_3 > x_2 > 0.$$

Thus we would have  $-(x_1 + x_2)(x_1 + x_3)(x_2 + x_3) < 0$ . But this contradicts the hypothesis. Therefore  $x_2 < 0, x_3 < 0$ .

Problem 744 gave another method of solution.

$$771 \quad s = \frac{1}{4} \sqrt{a(4ab - a^3 - 8c)}, \quad R = \frac{c}{\sqrt{a(4ab - a^3 - bc)}}.$$

$$772 \quad a(4ab - a^3 - 8c) = 4c^2.$$

$$773 \quad a) \frac{25}{27}; \quad b) \frac{35}{27}; \quad c) -\frac{1679}{625}.$$

$$774 \quad a) a_1^2 a_2^2 - 4a_1^3 a_3 - 4a_2^3 a_0 + 18a_0 a_1 a_2 a_3 - 27a_0^2 a_3^2;$$

$$b) a_1^3 a_3 - a_2^3 a_0; \quad c) \frac{a_1 a_2}{a_0 a_3} - 9; \quad d) a_1^2 a_2^2 - a_1^3 a_3 - a_2^3 a_0.$$

775 It is sufficient to give a proof for the elementary symmetric functions. Let  $\varphi_k$  be an elementary symmetric functions of degree  $k$  in  $x_2, x_3, \dots, x_n$ ;  $f_k$  an elementary symmetric function in  $x_1, x_2, \dots, x_n$ . Clearly,  $\varphi_k = f_k - x_1 \varphi_{k-1}$ . Thus:

$$\begin{aligned} \varphi_k &= f_k - x_1 f_{k-1} + x_1^2 f_{k-1} - \dots + (-x_1)^{k-1} f_1 + (-1)^k x_1^k, \\ (-1)^k \varphi_k &= a_k + a_{k-1} x_1 + \dots + a_1 x_1^{k-1} + x_1^k. \end{aligned}$$

$$\begin{aligned}
 776 \quad & x_1 + x_2 = f_1 - x_3; \\
 & (f_1 - x_1)(f_1 - x_2)(f_1 - x_3) = f_1^3 - f_1^3 + f_1 f_2 - f_3 = f_1 f_2 - f_3; \\
 & 2x_1 - x_2 - x_3 = 3x_1 - f_1; \\
 & (3x_1 - f_1)(3x_2 - f_1)(3x_3 - f_1) = 27f_3 - 9f_1 f_2 + 2f_1^3; \\
 & x_1^2 + x_1 x_2 + x_2^2 = f_1^2 - f_2 - f_1 x_3; \\
 & x_1^2 - x_2 x_3 = f_1 x_1 - f_2.
 \end{aligned}$$

$$777 \quad \sum_{i=1}^n \frac{\partial f_k}{\partial x_i} = (n-k) f_{k-1}.$$

778 Let  $F(x_1, x_2, \dots, x_n) = \Phi(f_1, f_2, \dots, f_n)$ . Then

$$\sum_{i=1}^n \frac{\partial F}{\partial x_i} = n \frac{\partial \Phi}{\partial f_1} + (n-1) f_1 \frac{\partial \Phi}{\partial f_2} + \dots + f_{n-1} \frac{\partial \Phi}{\partial f_n}.$$

779 Let  $\varphi(a) = F(x_1 + a, x_2 + a, \dots, x_n + a)$ . Then

$$\varphi'(a) = \sum_{i=1}^n \frac{\partial F(x_1 + a, x_2 + a, \dots, x_n + a)}{\partial x_i}.$$

Since  $\varphi(a)$  is independent of  $a$ , then  $\varphi'(a) \equiv 0$ ,  
 from which  $\sum_{i=1}^n \frac{\partial F}{\partial x_i} = 0$ . Conversely, if  
 $\sum_{i=1}^n \frac{\partial F(x_1, x_2, \dots, x_n)}{\partial x_i} \equiv 0$  then

$$\varphi'(a) = \sum_{i=1}^n \frac{\partial F(x_1 + a, \dots, x_n + a)}{\partial x_i} = 0 \quad \text{Therefore}$$

$\varphi(a)$  is independent of  $a$ ;  $\varphi(a) = \varphi(0)$ , i.e.

$$F(x_1 + a, x_2 + a, \dots, x_n + a) = F(x_1, x_2, \dots, x_n).$$

The result of the preceding problem shows that  
 the relation  $\sum_{i=1}^n \frac{\partial F}{\partial x_i} = 0$  is equivalent to the  
 condition

$$n \frac{\partial \Phi}{\partial f_1} + (n-1) f_1 \frac{\partial \Phi}{\partial f_2} + \dots + f_{n-1} \frac{\partial \Phi}{\partial f_n} = 0.$$

- 780 Let  $F(x_1, x_2, \dots, x_n)$  be a homogeneous symmetric function of degree 2. It must be expressible in terms of the elementary symmetric functions in the form  $F = Af_1^2 + Bf_2$ . By problem 779, we must have  $n2Af_1 + (n-1)Bf_1 = 0$ , so that

$$A = (n-1)\alpha, B = -2n\alpha, \text{ and}$$

$$F(x_1, x_2, \dots, x_n) = \alpha[(n-1)f_1^2 - 2nf_2] = \alpha \sum_{i < k} (x_i - x_k)^2.$$

- 781 A symmetric function of the third degree is the following combination of the elementary symmetric functions:  $Af_1^3 + Bf_1f_2 + Cf_3$ . By problem 779, the relation  $3Anf_1^2 + nBf_2 + (n-1)Bf_1^2 + (n-2)Cf_2 = 0$  must hold. Thus

$$F(x_1, x_2, \dots, x_n) = \alpha[(n-1)(n-2)f_1^3 - 3n(n-2)f_1f_2 + 3n^2f_3].$$

- 782  $(n-2)f_1^2f_2^2 - 2(n-1)f_1^3f_3 - 4(n-2)f_2^3$   
 $+ (10n-12)f_1f_2f_3 - 4(n-1)f_1^2f_4 - 9nf_3^2 + 8nf_2f_4.$

- 783 We may take

$$\varphi_k = f_k \left( x_1 - \frac{f_1}{n}, x_2 - \frac{f_1}{n}, \dots, x_n - \frac{f_1}{n} \right).$$

Each of the functions  $\varphi_k$  certainly has the required properties. Moreover if

$$F(x_1, x_2, \dots, x_n) = F(x_1 + a, x_2 + a, \dots, x_n + a), \text{ and}$$

$$F(x_1, x_2, \dots, x_n) = \Phi(f_1, f_2, \dots, f_n), \text{ then}$$

$$F(x_1, x_2, \dots, x_n) = \Phi(0, \varphi_2, \varphi_3, \dots, \varphi_n).$$

- 784 a)  $-4\varphi_2^3 - 27\varphi_3^2$ ; b)  $18\varphi_2^2$ .



- 785 a)  $8\varphi_3$ ;  
 b)  $-4\varphi_2^3\varphi_3^2 + 16\varphi_2^4\varphi_4 - 27\varphi_3^4 + 144\varphi_2\varphi_3^2\varphi_4 - 128\varphi_2^2\varphi_4^2 + 256\varphi_4^3$ .
- 786  $s_2 = f_1^2 - 2f_2$ ;  
 $s_3 = f_1^3 - 3f_1f_2 + 3f_3$ ;  
 $s_4 = f_1^4 - 4f_1^2f_2 + 2f_2^2 + 4f_1f_3 - 4f_4$ ;  
 $s_5 = f_1^5 - 5f_1^3f_2 + 5f_1f_2^2 + 5f_1^2f_3 - 5f_2f_3 - 5f_1f_4 + 5f_5$ ;  
 $s_6 = f_1^6 - 6f_1^4f_2 + 9f_1^2f_2^2 + 6f_1^3f_3 - 2f_2^3 - 12f_1f_2f_3 - 6f_1^2f_4 + 3f_3^2 + 6f_2f_4 + 6f_1f_5 - 6f_6$ .
- 787  $2f_2 = s_1^2 - s_2$ ;  
 $6f_3 = s_1^3 - 3s_1s_2 + 2s_3$ ;  
 $24f_4 = s_1^4 - 6s_1^2s_2 + 8s_1s_3 + 3s_2^2 - 6s_4$ ;  
 $120f_5 = s_1^5 - 10s_1^3s_2 + 20s_1^2s_3 + 15s_1s_2^2 - 20s_2s_3 - 30s_1s_4 + 24s_5$ ;  
 $720f_6 = s_1^6 - 15s_1^4s_2 + 40s_1^3s_3 + 45s_1^2s_2^2 - 120s_1s_2s_3 - 15s_2^3 - 90s_1^2s_4 + 40s_3^2 + 90s_2s_4 + 144s_1s_5 - 120s_6$ .
- 788  $s_5 = 859$ .
- 789  $s_8 = 13$ .
- 790  $s_{10} = 621$ .
- 791  $s_1 = -1, s_2 = s_3 = \dots = s_n = 0$ .
- 792 Easily established by induction, using the relation  

$$as_k + bs_{k-1} + cs_{k-2} = 0,$$
 where  $s_k = x_1^k + x_2^k$ .
- 793  $s_5 - s_1^5 = 5(f_1^2 - f_2)(f_3 - f_1f_2)$ ;  $s_3 - s_1^3 = 3(f_3 - f_1f_2)$ .
- 794  $s_5 = -5f_2f_3$ ;  $s_3 = 3f_3$ ;  $s_2 = -2f_2$ .
- 795  $s_7 = -7f_2f_5$ ;  $s_2 = -2f_2$ ;  $s_5 = 5f_5$ .
- 796  $x^n - a = 0$ .

$$797 \quad x^n - \frac{a}{1} x^{n-1} + \frac{a^2}{1 \cdot 2} x^{n-2} - \dots + (-1)^n \frac{a^n}{n!} = 0.$$

$$798 \quad x^n + \frac{P_1(\alpha)}{1} x^{n-1} + \frac{P_2(\alpha)}{2!} x^{n-2} + \dots + \frac{P_n(\alpha)}{n!} = 0 \quad , \text{ where}$$

$P_1, P_2, \dots, P_n$  are the Hermite polynomials

$$P_k(x) = (-1)^k e^{\frac{x^2}{2}} \frac{d^k e^{-\frac{x^2}{2}}}{dx^k},$$

and  $\alpha$  is a root of the Hermite polynomial

$P_{n+1}(x)$ .

Solution. Suppose the required equation has the form

$$x^n + \alpha x^{n-1} + a_2 x^{n-2} + \dots + a_n = 0.$$

By Newton's formulas,

$$\begin{aligned} a_1 &= \alpha, \\ 2a_2 &= \alpha a_1 - 1, \\ 3a_3 &= \alpha a_2 - a_1, \\ &\dots \dots \dots \\ k a_k &= \alpha a_{k-1} - a_{k-2}, \\ &\dots \dots \dots \\ n a_n &= \alpha a_{n-1} - a_{n-2}, \\ 0 &= \alpha a_n - a_{n-1}, \end{aligned}$$

These relations show that  $a_k$  is a polynomial of degree  $k$  in  $\alpha$ . Set  $k! a_k = P_k(\alpha)$ . Then, setting  $P_0 = 1$ , we obtain

$$P_1 = \alpha, \quad P_k - \alpha P_{k-1} + (k-1) P_{k-2} = 0; \quad -\alpha P_n + n P_{n-1} = 0.$$

The first two relations show that  $P_k$  is the Hermite polynomial in  $\alpha$  (see problem 707). The last relation gives  $P_{n+1}(\alpha) = 0$ .

$$799 \quad \frac{1}{2}(s_k^2 - s_{2k}).$$

$$800 \quad \sum_{i=1}^n (x + x_i)^k = \sum_{m=0}^k C_m^k s_{k-m} x^m;$$

$$\sum_{i=1}^n \sum_{j=1}^n (x_j + x_i)^k = \sum_{m=0}^k C_m^k s_{k-m} s_m;$$

$$\sum_{i < j} (x_i + x_j)^k = \frac{1}{2} \left( \sum_{m=0}^k C_m^k s_{k-m} s_m - 2^k s_k \right).$$

$$801 \quad \sum_{i < j} (x_i - x_j)^{2k} = \frac{1}{2} \sum_{m=0}^{2k} C_m^{2k} (-1)^m s_m s_{2k-m}.$$

802 Add to the first line  $-s_1$  times the second plus  $s_2$  times the third, ..., plus  $(-1)^{k-1} s_{k-1}$  times the  $k$ -th. By Newton's formula we obtain:

$$\det \begin{bmatrix} f_1 & 1 & 0 & \dots & 0 \\ 2f_2 & f_1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ (k-1)f_{k-1} & f_{k-2} & \dots & f_1 & 1 \\ kf_k & f_{k-1} & \dots & f_1 & \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & f_1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & f_{k-2} & f_{k-3} & \dots & 1 \\ (-1)^{k-1} s_k & f_{k-1} & \dots & f_1 & \end{bmatrix} = s_k.$$

803 Add to the first column  $-f_1$  times the second,  $f_2$  times the third, ...,  $(-1)^{k-1} f_{k-1}$  times the  $k$ -th. Use Newton's formula to establish the result.

$$804 \quad n!(x^n - f_1 x^{n-1} + f_2 x^{n-2} + \dots + (-1)^n f_n).$$

$$805 \quad \frac{\varphi(n)}{\varphi\left(\frac{n}{d}\right)} \mu\left(\frac{n}{d}\right), \text{ where } d = (m, n).$$

- 806 By problems 117, 119 it is sufficient to consider the case  $n = p_1 p_2 \dots p_k$  where  $p_1, p_2, \dots, p_k$  are distinct odd prime numbers. In this case  $s_1 = s_2 = s_4 = (-1)^k$ ;  $s_3 = 2(-1)^{k-1}$ , if  $3 \mid n$ ;  $s_3 = (-1)^k$  if  $3 \nmid n$ . Calculation with Newton's formula gives:

$$\begin{aligned} f_2 &= \frac{1 - (-1)^k}{2} ; \\ f_3 &= \frac{(-1)^{k-1} - 1}{2} , \text{ if } 3 \mid n; \\ f_3 &= \frac{(-1)^k - 1}{2} , \text{ if } 3 \nmid n; \\ f_4 &= \frac{(-1)^{k-1} - 1}{2} , \text{ if } 3 \mid n; \\ f_4 &= \frac{(-1)^{k-1} + 1}{2} , \text{ if } 3 \nmid n. \end{aligned}$$

- 807  $s_1 = s_2 = s_3 = \dots = s_n = a$ . Thus for  $k \leq n$
- $$\begin{aligned} k f_k &= a f_{k-1} - a f_{k-2} + \dots + (-1)^{k-1} f_1, \\ (k-1) f_{k-1} &= a f_{k-2} + \dots + (-1)^{k-2} f_1, \end{aligned}$$

from which

$$k f_k = (a - k + 1) f_{k-1}, \quad f_k = \frac{a - k + 1}{k} f_{k-1}.$$

It is obvious that  $f_1 = a$ . Therefore

$$f_2 = \frac{a(a-1)}{1 \cdot 2}, \dots, \quad f_k = \frac{a(a-1) \dots (a-k+1)}{1 \cdot 2 \dots k}.$$

And thus  $x_1, x_2, \dots, x_n$  are the roots of the equation

$$\begin{aligned} x^n - \frac{a}{1} x^{n-1} + \frac{a(a-1)}{1 \cdot 2} x^{n-2} - \dots + (-1)^n \frac{a(a-1) \dots (a-n+1)}{n!} &= 0; \\ s_{n+1} &= a - \frac{a(1-a)(2-a) \dots (n-a)}{n!}. \end{aligned}$$

$$\begin{aligned}
 808 \quad & (x-a)(x-b)[x^n + (a+b)x^{n-1} + \dots + (a^n + a^{n-1}b + \dots \\
 & \dots + b^n)] = (x-a)[x^{n+1} + ax^n + a^2x^{n-1} + \dots + a^nx \\
 & - b(a^n + a^{n-1}b + \dots + b^n)] = x^{n+2} - (a^{n+1} + a^nb + \dots + b^{n+1})x \\
 & + ab(a^n + a^{n+1} + \dots + b^n).
 \end{aligned}$$

In the new polynomials, the degrees of the sums

$\sigma_1, \sigma_2, \dots, \sigma_n$  are obviously zero. But

$\sigma_k = s_k + a^k + b^k$ . Therefore  $s_k = -(a^k + b^k)$  for  $1 \leq k \leq n$ .

$$809 \quad s_k = -a^k - b^k, \text{ if } k \text{ is odd;}$$

$$s_k = -\left(a^{\frac{k}{2}} - b^{\frac{k}{2}}\right)^2, \text{ if } k \text{ is even.}$$

$$\begin{aligned}
 810 \quad & \text{a) } (x+a)(x^2+ax+b)-c=0; \\
 & \text{b) } x(x-a^2+3b)^2-(a^2b^2-4a^3c-4b^3+18abc-27c^2)=0; \\
 & \text{c) } x^3+(3b-a^2)x^2+b(3b-a^2)x+b^3-a^3c=0; \\
 & \text{d) } x^2(x-a^2+3b)+(a^2b^2-4a^3c-4b^3+18abc-27c^2)=0; \\
 & \text{e) } x^3-(a^2-2b)x^2+(b^2-2ac)x-c^2=0; \\
 & \text{f) } x^3+(a^3-3ab+3c)x^2+(b^3-3abc+3c^2)x+c^3=0.
 \end{aligned}$$

$$811 \quad y^2 + (2a^3 - 9ab + 27c)y + (a^2 - 3b)^3 = 0.$$

$$812 \quad y^2 - \frac{ab-3c}{c}y + \frac{b^3+a^3c-6abc+9c^2}{c^2} = 0.$$

$$\begin{aligned}
 813 \quad & y^6 - \frac{ab-3c}{c}y^5 + \frac{b^3-5abc+6c^2}{c^2}y^4 - \\
 & - \frac{a^2b^2-2b^3-2a^3b+6abc-7c^2}{c^2}y^3 + \frac{b^3-5abc+6c^2}{c^2}y^2 - \frac{ab-3c}{c}y + 1 = 0.
 \end{aligned}$$

$$\begin{aligned}
 814 \quad & \text{a) } y^3 - by^2 + (ac - 4d)y - (a^2d + c^2 - 4bd) = 0 \\
 & \text{(Ferrari resolvent);}
 \end{aligned}$$

$$\begin{aligned} \text{b) } y^3 - (3a^2 - 8b)y^2 + (3a^4 - 16a^2b + 16b^2 \\ + 16ac - 64d)y - (a^3 - 4ab + 8c)^2 = 0 \\ \text{(Ferrari resolvent);} \end{aligned}$$

$$\text{c) } y^6 - by^5 + (ac - d)y^4 - (a^2d + c^2 - 2bd)y^3 + d(ac - d)y^2 - bd^2y + d^3 = 0;$$

$$\text{d) } y^6 + 3ay^5 + (3a^2 + 2b)y^4 + (a^3 + 4ab)y^3 + (2a^2b + b^2 + ac - 4d)y^2 + (ab^2 + a^2c - 4ad)y + (abc - a^2d - c^2) = 0.$$

$$815 \quad x = \frac{-a \pm \sqrt{a^2 - 4b + 4y_1} \pm \sqrt{a^2 - 4b + 4y_2} \pm \sqrt{a^2 - 4b + 4y_3}}{4}.$$

The ambiguous signs must be chosen so that the product of the square roots is  $-a^3 + 4ab - 8c$ .

$$816 \quad x = \frac{\pm \sqrt{4a + \sqrt[3]{b^2 - 64a^3}} \pm \sqrt{4a + \epsilon \sqrt[3]{b^2 - 64a^3}} \pm \sqrt{4a + \epsilon^2 \sqrt[3]{b^2 - 64a^3}}}{2},$$

$\epsilon = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ . Choose the ambiguous signs so that the product of the roots is  $-b$ .

$$817 \quad (y + a)^4(y^2 + 6ay + 25a^2) + 3125b^4y = 0.$$

Method. The roots of the equation we are seeking must be:

$$\begin{aligned} y_1 &= (x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1) \\ &\quad \times (x_1x_3 + x_3x_5 + x_5x_2 + x_2x_4 + x_4x_1); \\ y_2 &= (x_1x_3 + x_3x_2 + x_2x_5 + x_5x_4 + x_4x_1) \\ &\quad \times (x_1x_2 + x_2x_4 + x_4x_3 + x_3x_5 + x_5x_1); \\ y_3 &= (x_5x_2 + x_2x_4 + x_4x_3 + x_3x_1 + x_1x_5) \\ &\quad \times (x_5x_4 + x_4x_1 + x_1x_2 + x_2x_3 + x_3x_5); \\ y_4 &= (x_2x_1 + x_1x_3 + x_3x_5 + x_5x_4 + x_4x_2) \\ &\quad \times (x_2x_3 + x_3x_4 + x_4x_1 + x_1x_5 + x_5x_2); \\ y_5 &= (x_5x_3 + x_3x_2 + x_2x_4 + x_4x_1 + x_1x_5) \\ &\quad \times (x_5x_2 + x_2x_1 + x_1x_3 + x_3x_4 + x_4x_5); \\ y_6 &= (x_2x_1 + x_1x_4 + x_4x_3 + x_3x_5 + x_5x_2) \\ &\quad \times (x_2x_4 + x_4x_5 + x_5x_1 + x_1x_3 + x_3x_2). \end{aligned}$$

The the equation we are looking for must have the form

$$y^6 + c_1 a y^5 + c_2 a^2 y^4 + c_3 a^3 y^3 + c_4 a^4 y^2 + (c_5 a^5 + c_6 b^4) y + (c_7 a^6 + c_8 a b^4) = 0,$$

where  $c_1, c_2, \dots, c_8$  are absolute constants. To determine them we set  $a = -1, b = 0; a = 0, b = -1$ . We obtain the following table.

$a$	$b$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$y_1$	$y_2$	$y_3$	$y_4$	$y_5$	$y_6$
-1	0	1	$i$	-1	$-i$	0	1	$3-4i$	1	1	$3+4i$	1
0	-1	1	$\epsilon$	$\epsilon^2$	$\epsilon^3$	$\epsilon^4$	0	-5	$-5\epsilon^4$	$-5\epsilon^3$	$-5\epsilon^2$	$-5\epsilon$

In the first case the equation we seek must have the form:

$$(y-1)^4(y^2-6y+25)=0;$$

in the second case  $y^6+3125y=0$ . In this way we have determined all the coefficients except  $c_8$ . To show that  $c_8 = 0$ , we can take for example  $a = -5, b = 4$ . In this case  $x_1 = x_2 = 1$ , and the remaining roots satisfy the equation  $x^3 + 2x^2 + 3x + 4 = 0$ , and all the required calculations can be completed without difficulty.

818 Set  $f(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$ , where  $x_1, x_2, \dots, x_n$  are independent variables. Now set

$$x^{k-1}\varphi(x) = f(x)q_k(x) + r_k(x),$$

$$r_k(x) = c_{k1} + c_{k2}x + \dots + c_{kn}x^{n-1}.$$

It is obvious that the coefficients  $c_{k,s}$  are certain polynomials in  $x_1, x_2, \dots, x_n$ .

Therefore,

$$\begin{aligned} \det \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix} &= \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{bmatrix} = \\ &= \det \begin{bmatrix} r_1(x_1) & r_1(x_2) & \dots & r_1(x_n) \\ r_2(x_1) & r_2(x_2) & \dots & r_2(x_n) \\ \cdot & \cdot & \cdot & \cdot \\ r_n(x_1) & r_n(x_2) & \dots & r_n(x_n) \end{bmatrix} = \\ &= \det \begin{bmatrix} \varphi(x_1) & \varphi(x_2) & \dots & \varphi(x_n) \\ x_1\varphi(x_1) & x_2\varphi(x_2) & \dots & x_n\varphi(x_n) \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{n-1}\varphi(x_1) & x_2^{n-1}\varphi(x_2) & \dots & x_n^{n-1}\varphi(x_n) \end{bmatrix} = \\ &= \varphi(x_1)\varphi(x_2) \dots \varphi(x_n) \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{bmatrix}. \end{aligned}$$

so that

$$\det \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix} = \varphi(x_1)\varphi(x_2) \dots \varphi(x_n) = R(f, \varphi).$$

This last equation is an identity among polynomials in the independent variables  $x_1, x_2, \dots, x_n$  and therefore is valid for every numerical value of these variables.



819 First note that every polynomial  $\psi_k(x)$  has degree  $n - k$ . Use the following abbreviation:

$$\begin{aligned} f_k(x) &= a_0 x^{k-1} + \dots + a_{k-1}; \\ \bar{f}_k(x) &= a_k x^{n-k} + \dots + a_n; \\ \varphi_k(x) &= b_0 x^{k-1} + \dots + b_{k-1}; \\ \bar{\varphi}_k(x) &= b_k x^{n-k} + \dots + b_n. \end{aligned}$$

Then

$$\begin{aligned} f(x) &= x^{n-k+1} f_k(x) + \bar{f}_k(x); \\ \varphi(x) &= x^{n-k+1} \varphi_k(x) + \bar{\varphi}_k(x); \\ \psi_k(x) &= f_k(x) [x^{n-k+1} \varphi_k(x) + \bar{\varphi}_k(x)] - \\ &\quad - \varphi_k(x) [x^{n-k+1} f_k(x) + \bar{f}_k(x)] = f_k(x) \bar{\varphi}_k - \varphi_k \bar{f}_k(x) = \\ &\quad = (a_0 b_k - b_0 a_k) x^{n-1} + \dots \end{aligned}$$

Set  $\psi_k(x) = c_{k1} + c_{k2}x + \dots + c_{kn}x^{n-1}$  and let  $x_1, x_2, \dots, x_n$  be the roots of  $f(x)$ . Then

$$\begin{aligned} \det \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{bmatrix} = \\ = \det \begin{bmatrix} \psi_1(x_1) & \psi_1(x_2) & \dots & \psi_1(x_n) \\ \psi_2(x_1) & \psi_2(x_2) & \dots & \psi_2(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_n(x_1) & \psi_n(x_2) & \dots & \psi_n(x_n) \end{bmatrix} = \\ = \det \begin{bmatrix} f_1(x_1)\varphi(x_1) & f_1(x_2)\varphi(x_2) & \dots & f_1(x_n)\varphi(x_n) \\ f_2(x_1)\varphi(x_1) & f_2(x_2)\varphi(x_2) & \dots & f_2(x_n)\varphi(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ f_n(x_1)\varphi(x_1) & f_n(x_2)\varphi(x_2) & \dots & f_n(x_n)\varphi(x_n) \end{bmatrix} = \\ = \varphi(x_1)\varphi(x_2) \dots \varphi(x_n) \cdot \begin{bmatrix} f_1(x_1) & f_1(x_2) & \dots & f_1(x_n) \\ f_2(x_1) & f_2(x_2) & \dots & f_2(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ f_n(x_1) & f_n(x_2) & \dots & f_n(x_n) \end{bmatrix} = \end{aligned}$$

$$\begin{aligned}
 &= \varphi(x_1) \varphi(x_2) \dots \varphi(x_n) \\
 &\times \det \begin{bmatrix} a_0 & 0 & \dots & 0 \\ a_1 & a_0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{n-1} & a_{n-2} & \dots & a_0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{bmatrix} = \\
 &= \varphi(x_1) \varphi(x_2) \dots \varphi(x_n) a_0^n \cdot \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{bmatrix}.
 \end{aligned}$$

Thus we obtain

$$\det \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix} = a_0^n \varphi(x_1) \varphi(x_2) \dots \varphi(x_n) = R(f, \varphi).$$

820 The polynomials  $\chi_k$  have degree  $n - 1$  or less.

This is clear for  $1 \leq k \leq n - m$ , and for

$k > n - m$ , it follows from the fact that  $\chi_k$

is the Bezout polynomial  $\psi_{k-n+m}$  of  $f(x)$ ,  $x^{n-m}\varphi(x)$ .

Set  $\chi_k(x) = c_{k1} + c_{k2}x + \dots + c_{kn}x^{n-1}$ ;

$$\Delta = \det \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \dots & \dots & \dots & \dots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{bmatrix}.$$

Then

$$\begin{aligned}
 \Delta &= \det \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix} = \det \begin{bmatrix} \chi_1(x_1) & \chi_1(x_2) & \dots & \chi_1(x_n) \\ \chi_2(x_1) & \chi_2(x_2) & \dots & \chi_2(x_n) \\ \dots & \dots & \dots & \dots \\ \chi_n(x_1) & \chi_n(x_2) & \dots & \chi_n(x_n) \end{bmatrix} = \\
 &= \varphi(x_1) \varphi(x_2) \dots \varphi(x_n) \\
 &\det \begin{bmatrix} 1 & & & 1 & & & \dots & 1 \\ x_1 & & & x_2 & & & \dots & x_n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x_1^{n-m-1} & & & x_2^{n-m-1} & & & \dots & x_n^{n-m-1} \\ x_1^{n-m} f_0(x_1) & & & x_2^{n-m} f_0(x_2) & & & \dots & x_n^{n-m} f_0(x_n) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x_1^{n-m} f_{m-1}(x_1) & & & x_2^{n-m} f_{m-1}(x_2) & & & \dots & x_n^{n-m} f_{m-1}(x_n) \end{bmatrix} = \\
 &= \varphi(x_1) \varphi(x_2) \dots \varphi(x_n) \Delta \cdot \begin{vmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & \ddots & & & & \\ & & & & 1 & & & \\ & & & & & a_0 & & \\ & & & & & a_1 & a_0 & \\ & & & & & \dots & \dots & \dots \\ & & & & & a_{m-1} & a_{m-2} & \dots & a_0 \end{vmatrix} = \\
 &= a_0^m \varphi(x_1) \dots \varphi(x_n) \Delta,
 \end{aligned}$$

from which the required result follows.

821 a)  $-7$ ; b)  $243$ ; c)  $0$ ; d)  $-59$ ; e)  $4854$ ; f)  $(b_0a_2 - b_2a_0)^2 - (b_0a_1 - b_1a_0)(b_1a_2 - b_2a_1)$ .

822 a) For  $\lambda = 3$  and  $\lambda = -1$ ;

b)  $\lambda = 1, \quad \lambda = \frac{-2 + \sqrt{2} \pm \sqrt{4\sqrt{2} - 2}}{2},$

$\lambda = \frac{-2 - \sqrt{2} \pm i\sqrt{4\sqrt{2} + 2}}{2};$

c)  $\lambda = \pm \sqrt{-2}, \quad \lambda = \pm \sqrt{-12}.$

823 a)  $y^6 - 4y^4 + 3y^2 - 12y + 12 = 0$ ;

b)  $5y^5 - 7y^4 + 6y^3 - 2y^2 - y - 1 = 0$ ;

c)  $y^3 + 4y^2 - y - 4 = 0.$

824 a)  $x_1 = 1, \quad x_2 = 2, \quad x_3 = 0, \quad x_4 = -2,$

$y_1 = 2; \quad y_2 = 3; \quad y_3 = -1; \quad y_4 = 1.$

b)  $x_1 = 0, \quad x_2 = 3, \quad x_3 = 2, \quad x_4 = 2,$

$y_1 = 1; \quad y_2 = 0; \quad y_3 = 2; \quad y_4 = -1.$

c)  $x_1 = 1, \quad x_2 = 1, \quad x_3 = -1, \quad x_4 = 2,$

$y_1 = -1; \quad y_2 = -1; \quad y_3 = 1; \quad y_4 = 2.$

d)  $x_1 = 0, \quad x_2 = 0, \quad x_3 = 1, \quad x_4 = 1, \quad x_{5,6} = 2,$

$y_1 = 1; \quad y_2 = 3; \quad y_3 = 2; \quad y_4 = 3; \quad y_{5,6} = 1 \pm i\sqrt{2}.$

e)  $x_1 = 0, \quad x_2 = 0, \quad x_3 = 2, \quad x_4 = x_5 = 2, \quad x_6 = -4,$

$y_1 = 2; \quad y_2 = -2; \quad y_3 = 0; \quad y_4 = y_5 = 2; \quad y_6 = 2;$

$x_7 = 4, \quad x_8 = -6, \quad x_9 = -2/3,$

$y_7 = 6; \quad y_8 = 4; \quad y_9 = 4/3.$

825  $a_0^n a_n^{n-1}.$

826 Set  $f(x) = a_0(x - x_1)(x - x_2) \dots (x - x_n);$

$\varphi_1(x) = b_0x^k + \dots + b_k; \quad \varphi_2(x) = c_0x^m + \dots + c_m.$

Then

$$\begin{aligned} R(f, \varphi_1 \cdot \varphi_2) &= a_0^{m+k} \prod_{i=1}^n \varphi_1(x_i) \varphi_2(x_i) = \\ &= \left[ a_0^m \prod_{i=1}^n \varphi_1(x_i) \right] \left[ a_0^k \prod_{i=1}^n \varphi_2(x_i) \right] = R(f, \varphi_1) \cdot R(f, \varphi_2). \end{aligned}$$

827 Only the case  $n > 2$  is of interest. Set

$d = (m, n)$ ; let  $\xi_1, \xi_2, \dots$  be the primitive  $n$ -th roots of unity;  $\eta_1, \eta_2, \dots$  be the primitive  $n/d$ -th roots of unity;  $n_1 = n/d$ . Then

$$\begin{aligned} R(X_n, x^m - 1) &= \prod (\xi_i^m - 1) = \prod (1 - \xi_i^m) = \\ &= \left[ \prod (1 - \eta_i) \right]^{\frac{\varphi(n)}{\varphi(n_1)}} [X_{n_1}(1)]^{\frac{\varphi(n)}{\varphi(n_1)}}. \end{aligned}$$

If  $m$  is divisible by  $n$ , then  $R(X_n, x^m - 1) = 0$ .

If  $m$  is not divisible by  $n$ , then  $n_1 \neq 1$ , and by problem 123,  $X_{n_1}(1) = 1$  for  $X_{n_1}(1) = 1$ ;  $n_1 \neq p^\lambda$ ,  $X_{n_1}(1) = p$  for  $n_1 = p^\lambda$ , that is for  $n_1$  a prime power. Thus

$$R(X_n, x^m - 1) = 0 \quad \text{for } n_1 = n/d = 1;$$

$$R(X_n, x^m - 1) = p^{\frac{\varphi(n)}{\varphi(n_1)}} \quad \text{for } n_1 = n/d = p^\lambda;$$

$$R(X_n, x^m - 1) = 1 \quad \text{otherwise.}$$

828 It is clear that  $R(X_n, X_m)$  is a positive integer dividing  $R(X_n, x^m - 1)$  and  $R(X_m, x^n - 1)$ .

Set  $d = (m, n)$ . First suppose  $m \nmid n$ ,  $n \nmid m$ . Then  $m/d$ ,  $n/d$  are different from 1 and are relatively

prime. By the preceding problem  $R(X_n, x^m - 1)$ ,  $R(X_m, x^n - 1)$  are relatively prime and therefore  $R(X_n, X_m) = 1$ .

Now suppose that  $m \mid n$ . If  $m = n$ , then  $R(X_m, X_n) = 0$ . If  $m/n$  is not a prime power, then  $R(X_m, x^n - 1) = 1$ , and therefore  $R(X_m, X_n) = 1$ . Finally suppose that  $m = np^\lambda$ . Then

$$R(X_m, X_n) = \prod_{\delta \mid n} R(X_m, x^\delta - 1)^{\mu\left(\frac{n}{\delta}\right)}.$$

All the factors in this formula are 1 except for those for which  $m/\delta$  is a power of the prime  $p$ .

If  $n$  is not divisible by  $p$ , then a single factor (the one for  $\delta = n$ ) is different from 1, and

$$R(X_m, X_n) = R(X_m, x^n - 1) = p^{\frac{\varphi(m)}{\varphi(m/n)}} = p^{\varphi(n)}.$$

If  $n$  is divisible by  $p$ , then there are two factors different from 1: the factor for  $\delta = n$ , and the factor for  $\delta = n/p$ . Then

$$\begin{aligned} R(X_m, X_n) &= \frac{R(X_m, x^n - 1)}{R(X_m, x^{n/p} - 1)} = p^{\frac{\varphi(m)}{\varphi(m/n)} - \frac{\varphi(m)}{\varphi(m p/n)}} = \\ &= p^{\varphi(m) \left[ \frac{1}{p^{\lambda-1}(p-1)} - \frac{1}{p^\lambda(p-1)} \right]} = p^{\frac{\varphi(m)}{p^\lambda}} = p^{\varphi(n)}. \end{aligned}$$

Thus

$$\begin{aligned} R(X_m, X_n) &= 0 && \text{for } m = n; \\ R(X_m, X_n) &= p^{\varphi(n)} && \text{for } m = np^\lambda; \\ R(X_m, X_n) &= 1 && \text{otherwise.} \end{aligned}$$

829 a) 49; b) -107; c) -843; d) 725; e) 2777.

830 a)  $3125(b^2 - 4a^5)^2$ ; b)  $\lambda^4(4\lambda - 27)^3$ ;  
c)  $(b^2 - 3ab + 9a^2)^2$ ; d)  $4(\lambda^2 - 8\lambda + 32)^3$ .

831 a)  $\lambda = \pm 2$ ; b)  $\lambda_1 = 3$ ,  $\lambda_{2,3} = 3\left(-\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right)$ ;  
c)  $\lambda_1 = 0$ ,  $\lambda_2 = -3$ ,  $\lambda_3 = 125$ ;  
d)  $\lambda_1 = -1$ ,  $\lambda_2 = -\frac{3}{2}$ ,  $\lambda_{3,4} = \frac{7}{2} \pm \frac{9}{2}i\sqrt{3}$ .

832 In general if the discriminant is positive, the number of pairs of conjugate complex roots is even; if the discriminant is negative, the number of such pairs is odd.

In particular for a third degree polynomial all roots are real if  $D > 0$ ; there are two complex conjugate roots if  $D < 0$ .

If  $D > 0$  for a fourth degree polynomial, then either all roots are real or all roots are complex. For  $D < 0$  in this case, there are two real roots and a single pair of conjugate complex roots.

833  $f = x^n + a$ ;  $f' = nx^{n-1}$ ;  
 $R(f', f) = n^n a^{n-1}$ ;  $D(f) = (-1)^{\frac{n(n-1)}{2}} n^n a^{n-1}$ .

834  $f = x^n + px + q$ ;  $f' = nx^{n-1} + p$ ;  
 $R(f', f) = n^n \prod_{k=0}^{n-2} \left( q + \frac{n-1}{n} p \sqrt[n-1]{-\frac{p}{n} \epsilon^k} \right),$

where  $\epsilon = \cos \frac{2\pi}{n-1} + i \sin \frac{2\pi}{n-1}$ .

$$\begin{aligned} R(f', f) &= n^n \left[ q^{n-1} + \frac{(n-1)^{n-1} p^{n-1}}{n^{n-1}} \cdot \left( -\frac{p}{n} \right) (-1)^{n-2} \right] = \\ &= n^n q^{n-1} + (-1)^{n-1} (n-1)^{n-1} p^n; \\ D(f) &= (-1)^{\frac{n(n-1)}{2}} n^n q^{n-1} + (-1)^{\frac{(n-1)(n-2)}{2}} (n-1)^{n-1} p^n. \end{aligned}$$

835 Set  $d = (m, n)$ . Set  $m_1 = m/d$ ,  $n_1 = n/d$ , and let

$\epsilon$  be a primitive  $n$ -th root of 1,  $\eta$  a primitive

$n_1$ -th root of 1,  $f(x) = a_0 x^{m+n} + a_1 x^m + a_2$ .

Then  $f'(x) = (m+n) a_0 x^{m+n-1} + m a_1 x^{m-1}$ .

The roots of this derivative are:  $\xi_1 = \xi_2 = \dots = \xi_{m-1} = 0$ ,

$$\xi_{m+k} = \sqrt[n]{-\frac{m a_1}{(m+n) a_0}} \epsilon^k = \xi_m \epsilon^k, \quad k = 0, 1, \dots, n-1.$$

Thus

$$\begin{aligned} R(f', f) &= (m+n)^{m+n} a_0^{m+n} a_2^{m-1} \prod_{k=0}^{n-1} \left[ a_2 + \frac{n a_1}{m+n} \xi_m^m \epsilon^{km} \right] = \\ &= (m+n)^{m+n} a_0^{m+n} a_2^{m-1} \left[ \prod_{k=0}^{n_1-1} \left( a_2 + \frac{n a_1}{m+n} \right) \xi_m \eta^k \right]^d = \\ &= (m+n)^{m+n} a_0^{m+n} a_2^{m-1} \left[ a_2^{n_1} + (-1)^{m_1+n_1-1} \frac{n^{n_1} m^{m_1} a_1^{m_1+n_1}}{(m+n)^{m_1+n_1} a_0^{m_1}} \right]^d = \\ &= a_0^n a_2^{m-1} \left[ (m+n)^{m_1+n_1} a_0^{m_1} a_2^{n_1} + (-1)^{m_1+n_1-1} n^{n_1} m^{m_1} a_1^{m_1+n_1} \right]^d \end{aligned}$$

and therefore

$$\begin{aligned} D(f) &= (-1)^{\frac{(m+n)(m+n-1)}{2}} a_0^{n-1} a_2^{m-1} \left[ (m+n)^{m_1+n_1} a_0^{m_1} a_2^{n_1} + \right. \\ &\quad \left. + (-1)^{m_1+n_1-1} n^{n_1} m^{m_1} a_1^{m_1+n_1} \right]^d. \end{aligned}$$

836 The discriminants are equal.

837  $x_1 x_2 + x_3 x_4 - x_1 x_3 - x_2 x_4 = (x_1 - x_4)(x_2 - x_3);$

$x_1 x_3 + x_2 x_4 - x_1 x_4 - x_2 x_3 = (x_1 - x_2)(x_3 - x_4);$

$x_1 x_4 + x_2 x_3 - x_1 x_2 - x_3 x_4 = (x_1 - x_3)(x_4 - x_2).$



To obtain the desired conclusion, we square each of these formulas and multiply the results.

838 Set  $f(x) = a_0(x - x_1)(x - x_2) \dots (x - x_n)$ . Then

$$\begin{aligned} D(f(x)(x - a)) &= a_0^{2n} (a - x_1)^2 (a - x_2)^2 \dots (a - x_n)^2 \prod_{i < k} (x_i - x_k)^2 \\ &= D(f(x)) [f(a)]^2. \end{aligned}$$

839 Set  $\varphi(x) = x^{n-1} + x^{n-2} + \dots + 1$ . Then

$(x-1)\varphi(x) = x^n - 1$ , so that

$$D(\varphi) [\varphi(1)]^2 = D(x^n - 1) = (-1)^{\frac{(n-1)(n-2)}{2}} n^n.$$

Therefore

$$D(\varphi) = (-1)^{\frac{(n-1)(n-2)}{2}} n^{n-2}.$$

840 Set  $\varphi(x) = x^n + ax^{n-1} + ax^{n-2} + \dots + a$ . Then

$\varphi(x)(x-1) = x^{n+1} + (a-1)x^n - a$ . Thus

$$(na+1)^2 D(\varphi) = (-1)^{\frac{n(n-1)}{2}} a^{n-1} [(n+1)^{n+1} a + n^n (1-a)^{n+1}].$$

Finally

$$D(\varphi) = (-1)^{\frac{n(n-1)}{2}} a^{n-1} \frac{(n+1)^{n+1} a + n^n (1-a)^{n+1}}{(1+na)^2}.$$

841 Set  $f(x) = a_0(x - x_1)(x - x_2) \dots (x - x_n)$ ,

$$\varphi(x) = b_0(x - y_1)(x - y_2) \dots (x - y_m).$$

Then

$$\begin{aligned} D(f\varphi) &= (a_0 b_0)^{2m+2n-2} \prod_{i < k} (x_i - x_k)^2 \prod_{i < k} (y_i - y_k)^2 \prod_{i=1}^n \prod_{k=1}^m (x_i - y_k)^2 = \\ &= a_0^{2n-2} \prod_{i < k} (x_i - x_k)^2 b_0^{2m-2} \times \\ &\quad \times \prod_{i < k} (y_i - y_k)^2 \left[ a_0^m b_0^n \prod_{i=1}^n \prod_{k=1}^m (x_i - y_k) \right]^2 = D(f) D(\varphi) [R(f, \varphi)]^2. \end{aligned}$$

842  $X_{p^m}(x^{p^{m-1}} - 1) = x^{p^m} - 1$ . Thus

$$D(X_{p^m}) D(x^{p^{m-1}} - 1) [R(x^{p^{m-1}} - 1, X_{p^m})]^2 = D(x^{p^m} - 1).$$

Using known values in this relationship, we obtain

$$D(X_p^m) = p^m p^{m-(m+1)} p^{m-1} (-1)^{\frac{1}{2} p^{m-1} (p-1)}.$$

$$843 \quad X_n \prod_{\delta/n} (x^\delta - 1)^{\mu(\frac{n}{\delta})} = (x^n - 1) \prod_{\substack{\delta/n \\ \delta \neq n}} (x^\delta - 1)^{\mu(\frac{n}{\delta})}.$$

Let  $\epsilon$  be a root of  $X_n$ . Then

$$X'_n(\epsilon) = n \epsilon^{n-1} \prod_{\substack{\delta/n \\ \delta \neq n}} (\epsilon^\delta - 1)^{\mu(\frac{n}{\delta})}.$$

The calculation will be simplified if we take only the absolute value of the discriminant of  $X_n$ :

$$\begin{aligned} |D(X_n)| &= \prod_{\epsilon} |X'_n(\epsilon)| = n^{\varphi(n)} \prod_{\substack{\delta/n \\ \delta \neq n}} \prod_{\epsilon} (1 - \epsilon^\delta)^{\mu(\frac{n}{\delta})} = \\ &= n^{\varphi(n)} \prod_{\substack{\delta/n \\ \delta \neq n}} [X_n(1)]^{\frac{\varphi(n)}{\varphi(\frac{n}{\delta})} \mu(\frac{n}{\delta})}. \end{aligned}$$

Now,  $X_n(1) \neq 1$  only if  $n/\delta$  is a power of

some prime. On the other hand  $\mu(n/\delta) \neq 0$  only if  $n/\delta$  is not the square of a prime number.

Therefore the only factors to be retained are those corresponding to  $n/\delta = p_1, p_2, \dots, p_k$ , where  $p_1, p_2, \dots, p_k$  are the distinct prime divisors of  $n$ .

Thus

$$|D(X_n)| = \frac{n^{\varphi(n)}}{\prod_{p/n} p^{\varphi(n)/p-1}}.$$

Since all the roots of  $X_n$  are complex, the sign of its discriminant will be  $(-1)^{\frac{\varphi(n)}{2}}$ .

Thus

$$D(X_n) = (-1)^{\frac{\varphi(n)}{2}} \frac{n^{\varphi(n)}}{\prod_{p/n} p^{\varphi(n)/p-1}}.$$

$$844 \quad E_n = n! \left( 1 + \frac{x}{1} + \dots + \frac{x^n}{n!} \right),$$

$$E'_n = n! \left( 1 + \frac{x}{1} + \dots + \frac{x^{n-1}}{(n-1)!} \right).$$

Thus

$$E'_n = E_n - x^n;$$

$$R(E_n, E'_n) = \prod_{i=1}^n (-x_i)^n = (-1)^n [(-1)^n n!]^n = (n!)^n;$$

$$D(E_n) = (-1)^{\frac{n(n-1)}{2}} (n!)^n.$$

845 It is easy to see that

$$(nx + n - a)F_n - x(x+1)F'_n + \frac{a(a-1)\dots(a-n)}{n!} = 0.$$

Let  $x_1, x_2, \dots, x_n$  be the roots of  $F_n$ . Then

$$F'_n(x_i) = \frac{c}{x_i(x_i+1)}, \quad \text{where} \quad c = \frac{a(a-1)\dots(a-n)}{n!}.$$

Thus

$$R(F_n, F'_n) = \frac{c^n}{\prod x_i \prod (x_i + 1)} = \frac{c^n}{\frac{a(a-1)\dots(a-n+1)}{n!} \cdot \frac{(a-1)\dots(a-n)}{n!}} =$$

$$= \frac{a^{n-1}(a-1)^{n-2}(a-2)^{n-2}\dots(a-n+1)^{n-2}(a-n)^{n-1}}{(n!)^{n-2}};$$

$$D(F_n) = (-1)^{\frac{n(n-1)}{2}} \frac{a^{n-1}(a-1)^{n-2}(a-2)^{n-2}\dots(a-n+1)^{n-2}(a-n)^{n-1}}{(n!)^{n-2}}.$$

846  $P'_n = nP_{n-1}$ . Thus,

$$R(P_n, P'_n) = n^n R(P_n, P_{n-1}).$$

and

$$P_n - xP_{n-1} + (n-1)P_{n-2} = 0.$$

Therefore  $P_n(\xi) = -(n-1)P_{n-2}(\xi)$ , if  $\xi$  is a root of  $P_{n-1}$ , and therefore

$$\begin{aligned} R(P_n, P_{n-1}) &= (-1)^{n-1} (n-1)^{n-1} R(P_{n-2}, P_{n-1}) = \\ &= (-1)^{n-1} (n-1)^{n-1} R(P_{n-1}, P_{n-2}). \end{aligned}$$

Now it is easy to see that

$$R(P_n, P_{n-1}) = (-1)^{\frac{n(n-1)}{2}} (n-1)^{n-1} (n-2)^{n-2} \dots 2^2 \cdot 1.$$

Finally

$$D(P_n) = 1 \cdot 2^2 \cdot 3^3 \dots (n-1)^{n-1} n^n.$$

$$847. D(P_n) = 1 \cdot 2^3 \cdot 3^5 \dots n^{2n-1}.$$

$$848. D(P_n) = 2^{n-1} n^n.$$

$$849. D(P_n) = (n+1)^{n-1} \cdot 2^{n(n-1)}.$$

$$850. D(P_n) = 1 \cdot 2^3 \cdot 3^5 \dots n^{2n-1} \cdot 1^{2(n-1)} \cdot 3^{2(n-2)} \dots (2n-3)^2.$$

$$851. D(P_n) = 2^2 \cdot 3^4 \dots n^{2n-2} \cdot (n+1)^{n-1}.$$

$$\begin{aligned} 852 \quad \text{Set } f(x) &= x^n + a_1 x^{n-1} + \dots + a_n = \\ &= (x-x_1)(x-x_2) \dots (x-x_n), \end{aligned}$$

$D(f) = \prod (x_i - x_k)^2$ . We obtain the maximum of  $D(f)$  by solving the following system of equations

$$\begin{aligned} x_1^2 + x_2^2 + \dots + x_n^2 &= n(n-1)R^2, \\ \frac{\partial}{\partial x_i} (D - \lambda(x_1^2 + x_2^2 + \dots + x_n^2)) &= 0. \end{aligned}$$

It is easy to see that

$$\frac{\partial D}{\partial x_i} = \frac{Df''(x_i)}{f'(x_i)}.$$

Thus we have

$$f''(x) D - 2\lambda x_i f'(x_i) = 0 \quad \text{for } i = 1, 2, \dots, n.$$

Therefore the polynomial  $f(x)$  that has maximum discriminant must satisfy the differential

equation

$$cf(x) - 2\lambda xf'(x) + Df''(x) = 0,$$

where  $c$  is some constant. Dividing this by  $c/n$  and comparing coefficients of  $x^n$ , we see that the differential equation must have the form

$$nf(x) - xf'(x) + c'f''(x) = 0,$$

where  $c'$  is some new constant.

Using the method of undetermined coefficients for the next highest and next next highest powers of  $x$ , we find  $a_1 = 0$ ,  $a_2 = -n(n-1)c'/2$ . Now we can determine  $c'$ . Indeed

$$n(n-1)R^2 = x_1^2 + x_2^2 + \dots + x_n^2 = a_1^2 - 2a_2 = n(n-1)c'.$$

Thus  $c' = R^2$ .

Computing the remaining coefficients we find that  $f(x)$  has the form

$$f(x) = x^n - \frac{n(n-1)}{2} R^2 x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4} R^4 x^{n-4} - \dots$$

It is easy to see that

$$f(x) = R^n P_n\left(\frac{x}{R}\right),$$

where  $P_n$  is the Hermite polynomial. The discriminant is given by

$$D(f) = R^{n(n-1)} \cdot 1 \cdot 2^2 \cdot 3^3 \dots n^n.$$

This is the required maximum discriminant.

$$853 \quad 2^{2n} (-1)^n a_0 a_n [D(f)]^2.$$

$$854 \quad m^{mn} (-1)^{\frac{m(m-1)n}{2}} a_0^{m-1} a_n^{m-1} [D(f)]^m.$$

$$855 \quad F(x) = \prod_{i=1}^n (\varphi(x) - x_i).$$

Thus

$$D(F) = \prod_{i=1}^n D(\varphi(x) - x_i) \left[ \prod_{i < k} R(\varphi(x) - x_i, \varphi(x) - x_k) \right]^2.$$

It is clear therefore, that

$$R(\varphi(x) - x_i, \varphi(x) - x_k) = (x_i - x_k)^m.$$

Therefore

$$D(F) = \prod_{i=1}^n D(\varphi(x) - x_i) \prod_{i < k} (x_i - x_k)^{2m} = [D(f)]^m \prod_{i=1}^n D(\varphi(x) - x_i),$$

which established the assertion.

$$856 \quad (y + 1)(y = 5)(y - 19) = 0.$$

857 a) Solution.  $x^3 = 3x + 4$ . Set  $y = 1 + x + x^2$ , where  $x$  is a root of the given equation. Then

$$yx = x + x^2 + x^3 = x + x^2 + 3x + 4 = 4 + 4x + x^2;$$

$$yx^2 = 4x + 4x^2 + x^3 = 4x + 4x^2 + 3x + 4 = 4 + 7x + 4x^2.$$

Eliminating  $x$ , we obtain

$$\det \begin{bmatrix} 1-y & 1 & 1 \\ 4 & 4-y & 1 \\ 4 & 7 & 4-y \end{bmatrix} = 0,$$

$$y^3 - 9y^2 + 9y - 9 = 0;$$

$$b) y^3 - 7y^2 + 3y - 1 = 0;$$

$$c) y^4 + 5y^3 + 9y^2 + 7y - 6 = 0;$$

$$d) y^4 - 12y^3 + 43y^2 - 49y + 20 = 0.$$

$$858 \quad a) y^3 - 2y^2 + 6y - 4 = 0, \quad x = -\frac{y^2 - 2y + 4}{2};$$

$$b) y^4 - 9y^3 + 31y^2 - 45y + 13 = 0, \quad x = \frac{y^2 - 3y + 2}{3};$$

c)  $y^4 + 2y^3 - y^2 - 2y + 1 = 0$  ; no inverse transformation.

859  $y^3 - y^2 - 2y + 1 = 0.$

The transformed equation is the same as the original. This means that the original equation happens to have roots  $x_1, x_2$ , connected by the relation  $x_2 = 2 - x_1^2$ .

860  $x_2$  is expressible rationally in terms of  $x_1$ . Since  $x_1^3$  and higher powers of  $x_1$  can be expressed rationally in terms of lower powers, we may write  $x_2 = ax_1^2 + bx_1 + c$ . The numbers  $ax_1^2 + bx_1 + c, ax_2^2 + bx_2 + c, ax_3^2 + bx_3 + c$  are roots of some cubic equation with rational coefficients. One of these is a root of the given equation. All are rational, and the given equation is irreducible. Thus all three are roots of the given equation. In other words either  $ax_2^2 + bx_2 + c = x_3, ax_3^2 + bx_3 + c = x_1$ , or else  $ax_2^2 + bx_2 + c = x_1, ax_3^2 + bx_3 + c = x_3$ . The latter is eliminated by the remark that  $x_3$  cannot satisfy a second degree equation with rational coefficients. Thus we have the following:

$$x_2 = ax_1^2 + bx_1 + c;$$

$$x_3 = ax_2^2 + bx_2 + c;$$

$$x_1 = ax_3^2 + bx_3 + c.$$

Therefore

$$\begin{aligned} \sqrt{D} &= (x_2 - x_1)(x_3 - x_2)(x_1 - x_3) = \\ &= [ax_1^2 + (b-1)x_1 + c][ax_2^2 + (b-1)x_2 + c][ax_3^2 + (b-1)x_3 + c] \end{aligned}$$

is a rational number since it is a symmetric function of the roots  $x_1, x_2, x_3$  with rational coefficients. This shows the necessity of the given condition.

Now suppose that the discriminant  $D$  is the square of a rational number  $d$ . Then

$$x_2 - x_3 = \frac{d}{(x_1 - x_2)(x_1 - x_3)} = \frac{d}{3x_1^2 + 2ax_1 + b}.$$

On the other hand,

$$x_2 + x_3 = -a - x_1.$$

This shows that  $x_2, x_3$  are rational functions of  $x_1$ . The condition has been proved necessary.

861 a)  $\frac{2 + \sqrt{2} + \sqrt{6}}{4}$ ; b)  $\frac{-3 + 7\sqrt[3]{2} - \sqrt[3]{4}}{23}$ ;  
c)  $1 + 3\sqrt[4]{2} + 2\sqrt{2} - \sqrt[4]{8}$ .

862 a)  $\frac{\alpha^2 - \alpha + 1}{3}$ ; b)  $17\alpha^2 - 3\alpha + 55$ ; c)  $3 - 10\alpha + 8\alpha^2 - 3\alpha^3$ ;

d) the denominator is 0 for  $\alpha$  equal to one of the roots in the equation.

863  $mx_1^2 + nx_1 + p = \frac{(pm - bm^2 + amn - n^2)x_1 + (amp - np - cm^2)}{mx_1 + ma - n}.$

864 If  $x_2 = \frac{\alpha x_1 + \beta}{\gamma x_1 + \delta}$ , then  $x_3 = \frac{\alpha x_2 + \beta}{\gamma x_2 + \delta}$

and

$$x_1 = \frac{\alpha x_3 + \beta}{\gamma x_3 + \delta} = \frac{(\alpha^2 + \beta\gamma)x_2 + (\alpha + \delta)\beta}{(\alpha + \delta)\gamma x_2 + (\beta\gamma + \delta^2)}.$$



On the other hand  $x_1 = \frac{-\delta x_2 + \beta}{\gamma x_2 - \alpha}$ , which shows the necessity of the relation

$$\alpha\delta - \beta\gamma = (\alpha + \delta)^2.$$

865 Suppose

$$a_0x^n + a_1x^{n-1} + \dots + a_n = a_0(x - x_1)(x - x_2)\dots(x - x_n).$$

Then

$$a_0x^n - a_1x^{n-1} + \dots + (-1)^n a_n = a_0(x + x_1)(x + x_2)\dots(x + x_n).$$

Multiplying these equations we obtain

$$\begin{aligned} a_0^2(x^2 - x_1^2)(x^2 - x_2^2)\dots(x^2 - x_n^2) \\ = (a_0x^n + a_2x^{n-2} + \dots)^2 - (a_1x^{n-1} + a_3x^{n-3} + \dots)^2. \end{aligned}$$

Thus we see that, to carry out the transformation

$y = x^2$ , we must replace  $x^2$  by  $y$  in the equation

$$(a_0x^n + a_2x^{n-2} + \dots)^2 - (a_1x^{n-1} + a_3x^{n-3} + \dots)^2 = 0.$$

866 Replace  $x^3$  by  $y$  in the equation

$$\begin{aligned} (a_0x^n + a_3x^{n-3} + \dots)^3 + (a_1x^{n-1} + a_4x^{n-4} + \dots)^3 \\ + (a_2x^{n-2} + a_5x^{n-5} + \dots)^3 - 3(a_0x^n + a_3x^{n-3} + \dots) \\ \times (a_1x^{n-1} + a_4x^{n-4} + \dots)(a_2x^{n-2} + a_5x^{n-5} + \dots) = 0. \end{aligned}$$

867 If a polynomial  $x^n + a_1x^{n-1} + \dots$  has integral coefficients, and if the moduli of all its roots are bounded by 1, then the coefficients must be bounded:

$$|a_k| \leq \frac{n(n-1)\dots(n-k+1)}{k!}.$$

Therefore there is no more than a finite number of polynomials of degree  $n$  with this property.

Let  $f = x^n + a_1x^{n-1} + \dots + a_n$ ,  $a_n \neq 0$

be one such polynomial; let  $x_1, x_2, \dots, x_n$  be its

roots. Set  $f_m = (x - x_1^m)(x - x_2^m) \dots (x - x_n^m)$ . All the polynomials  $f_m$  so constructed will have integral coefficients and roots not exceeding 1 in modulus. Therefore the various polynomials  $f_m$ ,  $m = 1, 2, \dots$  must comprise only a finite number of different ones. Thus we can choose an infinite sequence  $m_0 < m_1 < m_2 < \dots$ , such that  $f_{m_0} = f_{m_1} = f_{m_2} = \dots$ . That is,

$$\begin{aligned} x_1^{m_i} &= x_{\alpha_1}^{m_0}, \\ x_2^{m_i} &= x_{\alpha_2}^{m_0}, \\ &\vdots \\ x_n^{m_i} &= x_{\alpha_n}^{m_0}, \end{aligned}$$

where  $(\alpha_1, \alpha_2, \dots, \alpha_n)$  is a rearrangement of the integers  $1, 2, \dots, n$ . As we have an infinite number of such relations, and as there is only a finite number of permutations on  $n$  symbols, we can find two (in fact an infinite number) of positive exponents  $m_{i_1}, m_{i_2}$ , that corresponds to the same permutation. Thus we can make the following assertion:

$$\begin{aligned} x_1^{m_{i_1}} &= x_1^{m_{i_2}}, \\ x_2^{m_{i_1}} &= x_2^{m_{i_2}}, \\ &\vdots \\ x_n^{m_{i_1}} &= x_n^{m_{i_2}}. \end{aligned}$$

Since  $a_n \neq 0$ , the numbers  $x_1, x_2, \dots, x_n$  are nonzero and are therefore all roots of unity; in fact correspond to the same exponent  $m_{i_2} - m_{i_1}$ .

- 868 Let  $F(x_1, x_2, \dots, x_n)$  be an alternating polynomial, that is, one that changes to its negative under an odd permutation of the variables. Since

$$F(x_2, x_2, x_3, \dots, x_n) = -F(x_2, x_2, \dots, x_n) = 0,$$

$F(x_1, x_2, \dots, x_n)$  must be divisible by  $x_1 - x_2$

and in general must be divisible by the product

$\Delta = \prod_{i>k} (x_i - x_k)$ . Since  $\Delta$  is also an alternating polynomial, the polynomial  $F/\Delta$  must be a symmetric polynomial.

- 869 Let the given polynomial be  $\varphi$ . We know that  $\varphi$  is invariant under an even permutation; the group of all permutations can change  $\varphi$  at most into two forms: even permutations leave it invariant; odd permutations change it to  $\bar{\varphi}$ . Moreover, every odd permutation must change  $\bar{\varphi}$  to  $\varphi$ . Now we note that  $\varphi + \bar{\varphi}$  is invariant under the entire permutation group;  $\varphi - \bar{\varphi}$  is changed to its negative by an every odd permutation. Therefore

$$\varphi = \frac{\varphi + \bar{\varphi}}{2} + \frac{\varphi - \bar{\varphi}}{2} = F_1 + F_2 \Delta.$$

See problem 868.

- 870  $(f_1^2 - f_2)\Delta$ , where  $f_1, f_2$  are elementary symmetric functions in  $x_1, x_2, \dots, x_n$

- 871  $u^3 + a(a + \beta + \gamma)u^2 + [(a^2 + \beta^2 + \gamma^2)b + (a\beta + a\gamma + \beta\gamma) \times (a^2 - b)]u + c(a^3 + \beta^3 + \gamma^3) + \frac{ab - 3c}{2}(a^2\beta + a\beta^2 + a^2\gamma + a\gamma^2 + \beta^2\gamma + \beta\gamma^2) + a\beta\gamma(a^3 - 3ab + 6c) + \frac{(a - \beta)(\beta - \gamma)(\gamma - a)}{2} \sqrt{\Delta} = 0,$

where  $\Delta$  is the discriminant of the given equation.

872  $u^3 - 3pp'u - \frac{27qq' + \sqrt{\Delta\Delta'}}{2} = 0$ , where  $\Delta, \Delta'$  are the discriminants of the given equations.

873 Let  $y = ax^2 + bx + c$  be the Tschirnhaus transformation connecting the given equations.

Then their roots can be numbered so that

$$x_1y_1 + x_2y_2 + x_3y_3 = a(x_1^3 + x_2^3 + x_3^3) + b(x_1^2 + x_2^2 + x_3^2) + c(x_1 + x_2 + x_3)$$

is a rational number. Therefore one of the equations

$$u^3 - 3pp'u - \frac{27qq' \pm \sqrt{\Delta\Delta'}}{2} = 0$$

must have rational roots (see problem 872).

Therefore  $\sqrt{\Delta\Delta'}$  must be a rational number. The condition is seen to be necessary.

Conversely, suppose that the equation

$$u^3 - 3pp'u - \frac{27qq' \pm \sqrt{\Delta\Delta'}}{2} = 0 \quad (*)$$

has a rational root  $u$ .

We compute the discriminant of the equation (\*) and find  $\frac{27^2}{4}(q\sqrt{\Delta'} - \sqrt{\Delta}q')^2$ . Since the ratio  $\Delta'/\Delta$  is the square of a rational number by hypothesis, the discriminant is  $\Delta$  times the square of a rational number. Now consider the other two roots,  $u', u''$ . The product  $u'u''$  is a rational number; so that the difference  $u' - u''$  can be expressed rationally in terms of the discriminant and  $u$ .

Thus  $u' - u''$  is a rational number times  $\sqrt{\Delta}$ .

The following system of equations is satisfied by

$y_1, y_2, y_3$ :

$$\begin{aligned} y_1 + y_2 + y_3 &= 0, \\ x_1 y_1 + x_2 y_2 + x_3 y_3 &= u, \\ (x_2 - x_3) y_1 + (x_3 - x_1) y_2 + (x_1 - x_2) y_3 &= u' - u'' = r \sqrt{\Delta}. \end{aligned}$$

From these equations we can obtain by elimination:

$$y_1 = \frac{-3ux_1 + (x_2 - x_3)r\sqrt{\Delta}}{6p}.$$

But  $(x_2 - x_3)\sqrt{\Delta}$  can be expressed rationally in terms of  $x_1$ . The condition is seen to be sufficient.

- 874 The variables  $x_1, x_2, \dots, x_n$  can be written as linear combinations in  $f_1, \eta_1, \eta_2, \dots, \eta_{n-1}$ . Therefore, any polynomial in  $x_1, x_2, \dots, x_n$  can be written as a polynomial in  $f_1, \eta_1, \eta_2, \dots, \eta_{n-1}$ :

$$F(x_1, x_2, \dots, x_n) = \sum A f_1^{\alpha_0} \eta_1^{\alpha_1} \eta_2^{\alpha_2} \dots \eta_{n-1}^{\alpha_{n-1}}.$$

A circular permutation of the variables  $x_1, x_2, \dots, x_n$  adds the factor  $\varepsilon^{-(\alpha_1 + 2\alpha_2 + \dots + (n-1)\alpha_{n-1})}$  to a typical term  $A f_1^{\alpha_0} \eta_1^{\alpha_1} \eta_2^{\alpha_2} \dots \eta_{n-1}^{\alpha_{n-1}}$ . Therefore a necessary condition that  $F(x_1, x_2, \dots, x_n)$  be invariant under all circular permutations is that  $\alpha_1 + 2\alpha_2 + \dots + (n-1)\alpha_{n-1}$  be divisible by  $n$ .

- 875 A possible choice is

$$f_1, \eta_1^n, \eta_2 \eta_1^{-2}, \dots, \eta_{n-1} \eta_1^{-(n-1)}.$$

876 Set  $\eta_1 = x_1 + x_2\varepsilon + x_3\varepsilon^2$ ;  $\eta_2 = x_1 + x_2\varepsilon^2 + x_3\varepsilon$ , where

$$\varepsilon = -\frac{1}{2} + i\frac{\sqrt{3}}{2}. \text{ Then } \frac{\eta_1^2}{\eta_2} = \varphi_1 + i\sqrt{3}\varphi_2, \text{ where}$$

$\varphi_1, \varphi_2$  are certain rational functions in

$x_1, x_2, x_3$  with rational coefficients, invariant under a circular permutation of  $x_1, x_2, x_3$ .

It remains to show that every rational function of  $x_1, x_2, x_3$  invariant under circular permutations can be expressed in terms of  $f_1 = x_1 + x_2 + x_3$ ,  $\varphi_1, \varphi_2$ . Indeed it is enough to show this for

$$\eta_2\eta_1^{-2}, \eta_1^3. \text{ But}$$

$$\eta_2\eta_1^{-2} = \frac{1}{\varphi_1 + i\varphi_2\sqrt{3}};$$

$$\eta_1^3 = \left(\frac{\eta_1^2}{\eta_2}\right)^2 \cdot \frac{\eta_2^2}{\eta_1} = (\varphi_1 + i\varphi_2\sqrt{3})^2(\varphi_1 - i\varphi_2\sqrt{3}).$$

877 For  $n = 4$ ,

$$\begin{aligned}\eta_1 &= x_1 + ix_2 - x_3 - ix_4, \\ \eta_2 &= x_1 - x_2 + x_3 - x_4, \\ \eta_3 &= x_1 - ix_2 - x_3 + ix_4.\end{aligned}$$

$$\text{Set } \theta_1 = \eta_1\eta_3; \quad \theta_2 + i\theta_3 = \frac{\eta_1\eta_2}{\eta_3}; \quad \theta_2 - i\theta_3 = \frac{\eta_3\eta_2}{\eta_1}.$$

Now  $\theta_1, \theta_2, \theta_3$  are rational functions of

$x_1, x_2, x_3, x_4$  with rational coefficients; they are invariant under circular permutations. As in problem 876 one sees that the function

$f = x_1 + x_2 + x_3 + x_4$  together with  $\theta_1, \theta_2, \theta_3$  form a basis for the functions needed. Indeed,

$$\eta_2\eta_1^{-2} = \frac{\theta_2 - i\theta_3}{\theta_1}; \quad \eta_3\eta_1^{-3} = \frac{\theta_2 - i\theta_3}{\theta_1(\theta_2 + i\theta_3)}; \quad \eta_1^4 = \frac{\theta_1^4(\theta_2 + i\theta_3)}{\theta_2 - i\theta_3}.$$

878 Set

$$\begin{aligned}
 \eta_1 &= x_1\varepsilon + x_2\varepsilon^2 + x_3\varepsilon^3 + x_4\varepsilon^4 + x_5, \\
 \eta_2 &= x_1\varepsilon^2 + x_2\varepsilon^4 + x_3\varepsilon + x_4\varepsilon^3 + x_5, \\
 \eta_3 &= x_1\varepsilon^3 + x_2\varepsilon + x_3\varepsilon^4 + x_4\varepsilon^2 + x_5, \\
 \eta_4 &= x_1\varepsilon^4 + x_2\varepsilon^3 + x_3\varepsilon^2 + x_4\varepsilon + x_5.
 \end{aligned}$$

Consider the rational function  $\lambda_1 = \frac{\eta_1\eta_2}{\eta_3}$ .

If we replace 1 by  $-\varepsilon - \varepsilon^2 - \varepsilon^3 - \varepsilon^4$ , we expand  $\lambda_1$  in powers of  $\varepsilon$ :

$$\lambda_1 = \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \varepsilon^3\varphi_3 + \varepsilon^4\varphi_4.$$

The coefficients  $\varphi_1, \varphi_2, \varphi_3, \varphi_4$  are rational numbers. We multiply the preceding equation by  $\varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4$ , and obtain

$$\begin{aligned}
 \lambda_2 &= \frac{\eta_2\eta_4}{\eta_1} = \varepsilon^2\varphi_1 + \varepsilon^4\varphi_2 + \varepsilon\varphi_3 + \varepsilon^3\varphi_4, \\
 \lambda_3 &= \frac{\eta_3\eta_1}{\eta_4} = \varepsilon^3\varphi_1 + \varepsilon\varphi_2 + \varepsilon^4\varphi_3 + \varepsilon^2\varphi_4, \\
 \lambda_4 &= \frac{\eta_4\eta_3}{\eta_2} = \varepsilon^4\varphi_1 + \varepsilon^3\varphi_2 + \varepsilon^2\varphi_3 + \varepsilon\varphi_4.
 \end{aligned}$$

The functions  $f, \varphi_1, \varphi_2, \varphi_3, \varphi_4$  form a basis for the functions invariant under circular permutations. This is shown by the following set of four equations:

$$\begin{aligned}
 \eta_2\eta_1^{-2} &= \lambda_1^{-1}\lambda_2\lambda_4^{-1}, \\
 \eta_3\eta_1^{-3} &= \lambda_1^{-2}\lambda_2\lambda_4^{-1}, \\
 \eta_4\eta_1^{-4} &= \lambda_1^{-2}\lambda_2\lambda_3^{-1}\lambda_4^{-1}, \\
 \eta_1^5 &= \lambda_1^3\lambda_2^{-1}\lambda_3\lambda_4^2.
 \end{aligned}$$

# CHAPTER VII - SOLUTIONS

## LINEAR ALGEBRA

- 879 a) Dimension  $r = 2$ ; possible basis vectors  $X_1, X_2$ ;  
 b) Dimension  $r = 2$ ; possible basis vectors  $X_1, X_2$ ;  
 c) Dimension  $r = 2$ ; possible basis vectors  $X_1, X_2$ .

- 880 a) The intersection has dimension 1 with basis vector

$$Z = (5, -2, -3, -4) = X_1 - 4X_2 = 3Y_1 - Y_2.$$

The sum has dimension 3 with possible basis  $Z, X_1, Y_1$ .

- b) The sum is the same as the first space; the intersection is the same as the second.

- c) The sum is the entire 4-dimensional space; the intersection is the null vector only.

- 881 a)  $\left(\frac{5}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$ ; b)  $(1, 0, -1, 0)$ .

- 882 a)  $x'_1 = \frac{x_1 + x_2 - x_3 - x_4}{2}, x'_2 = \frac{x_1 - x_2 + x_3 - x_4}{2},$   
 $x'_3 = \frac{x_1 - x_2 - x_3 + x_4}{2}, x'_4 = \frac{-x_1 + x_2 + x_3 + x_4}{2};$   
 b)  $x'_1 = x_2 - x_3 + x_4, x'_2 = -x_1 + x_2, x'_3 = x_4,$   
 $x'_4 = x_1 - x_2 + x_3 - x_4.$

- 883  $x'_1 x'_2 + x'_3 x'_4 = \frac{1}{8}.$

- 884 Let  $a_0 + a_1 \cos x + a_2 \cos^2 x + \dots + a_n \cos^n x =$   
 $b_0 + b_1 \cos x + b_2 \cos 2x + \dots + b_n \cos nx.$

Then  $a_0 = b_0 - b_2 + b_4 - \dots,$

$$a_k = 2^{k-1} [b_k + \sum_{1 \leq p \leq \frac{n-k}{2}} (-1)^p \frac{(k+2p)(k+p-1)(k+p-2) \dots (k+1)}{p!} b_{k+2p}].$$



$$b_0 = a_0 + \sum_{1 \leq p \leq \frac{n}{2}} 2^{-2p} C_p^{2p} a_{2p},$$

$$b_k = 2^{1-k} \left( a_k + \sum_{1 \leq p \leq \frac{n-k}{2}} 2^{-2p} C_p^{k+2p} a_{k+2p} \right).$$

- 885 For the first line the intersection point has coordinates  $(14/3, 1/9, 7/9, 11/9)$ . For the second line the intersection point is  $(42, 1, 7, 11)$ .
- 886 The lines  $X_0 + tX_1, Y_0 + tY_1$  lie in the manifold  $X_0 + t(Y_0 - X_0) + t_1X_1 + t_2Y_1$ .
- 887 In  $n$ -dimensional space the problem is solvable if and only if the lines  $X_0 + tX_1, Y_0 + tY_1$  are such that the vectors  $X_0, Y_0, X_1, Y_1$  are linearly dependent. This occurs if and only if the lines lie in a three-dimensional subspace through the origin.
- 888 The planes  $X_0 + t_1X_1 + t_2X_2, Y_0 + t_1Y_1 + t_2Y_2$  are subsets of the manifold  $X_0 + t(Y_0 - X_0) + t_1X_1 + t_2X_2 + t_3Y_1 + t_4Y_2$ .
- 889 There are 6 possible cases:
- 1) The planes have no common point and lie in no single 4-dimensional linear manifold (the planes are fully skewed).
  - 2) The planes have no common point but do lie in a 4-dimensional manifold but in no 3-dimensional manifold. (Skew-parallel planes.)

- 3) The planes have no common point but do lie in a 3-dimensional manifold (parallel planes).
- 4) The planes have a single common point. In this case they lie in a 4-dimensional manifold but in no 3-dimensional manifold.
- 5) The planes intersect in a line (have a common line).
- 6) The planes coincide.

If  $n = 3$  (in 3-dimensional space) only cases 3, 5, 6 can occur.

890 Let  $Q = X_0 + P$  be the linear manifold, where  $P$  is the linear space. If  $X_1 \in Q, X_2 \in Q$ , then  $X_1 = X_0 + Y_1, X_2 = X_0 + Y_2$ , where  $Y_1, Y_2$  contain  $P$ . Then for arbitrary  $\alpha$ , the linear combination  $\alpha X_1 + (1 - \alpha)X_2 = X_0 + \alpha Y_1 + (1 - \alpha)Y_2 \in Q$ . Conversely, let  $Q$  be a collection of vectors containing every linear combination  $\alpha X_1 + (1 - \alpha)X_2$  of every two factors  $X_1, X_2$  that it contains. Let  $X_0$  be some fixed vector in  $Q$ , let  $P$  denote the set of all vectors  $Y = X - X_0$ . If  $Y \in P$ , then  $cY \in P$  for every  $c$ , since  $cY = cX + (1 - c)X_0 - X_0$ . Moreover if  $Y_1 = X_1 - X_0 \in P, Y_2 = X_2 - X_0 \in P$ , then  $\alpha Y_1 + (1 - \alpha)Y_2 = \alpha X_1 + (1 - \alpha)X_2 - X_0 \in P$  for every  $\alpha$ . Now choose some fixed number  $\alpha, (\alpha \neq 0, \alpha \neq 1)$  and arbitrary  $c_1, c_2$ . Then  $(c_1/\alpha)Y_1 \in P, c_2/(1 - \alpha) \cdot Y_2 \in P$  for arbitrary  $Y_1, Y_2 \in P$ . Therefore also

$$c_1 Y_1 + c_2 Y_2 = \alpha \frac{c_1}{\alpha} Y_1 + (1 - \alpha) \frac{c_2}{1 - \alpha} Y_2 \in P.$$

Therefore  $P$  is a linear space,  $Q$  is a linear manifold.

Remark. The result is not valid for vector spaces over a field of characteristic 2.

891 a) 9; b) 0.

892 a)  $90^\circ$ ; b)  $45^\circ$ ; c)  $\cos \varphi = \frac{3}{\sqrt{77}}$ .

893  $\cos \varphi = \frac{1}{\sqrt{n}}$ .

894  $\cos A = \frac{5}{\sqrt{39}}$ ,  $\cos B = \frac{8}{\sqrt{78}}$ ,  $\cos C = -\frac{\sqrt{2}}{3}$ .

895  $\sqrt{n}$ .

896 If  $n$  is odd there are no orthogonal diagonals.  
If  $n = 2m$ , the number of orthogonal diagonals is  $C_{m-1}^{2m-1}$ .

897 The points have coordinates given by the rows of the matrix:

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{2} & \sqrt{\frac{3}{4}} & 0 & \dots & 0 & 0 \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \sqrt{\frac{4}{6}} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{24}} & \dots & \frac{1}{\sqrt{2n(n-1)}} & \sqrt{\frac{n+1}{2n}} \end{pmatrix}.$$

$$898 \quad R = \sqrt{\frac{n}{2(n+1)}}.$$

Coordinates of the center:

$$\left(\frac{1}{2}, \frac{1}{\sqrt{12}}, \dots, \frac{1}{\sqrt{2n(n-1)}}, \frac{1}{\sqrt{2(n+1)n}}\right).$$

$$899 \quad \left(\frac{3}{\sqrt{15}}, \frac{1}{\sqrt{15}}, \frac{2}{\sqrt{15}}, \frac{1}{\sqrt{15}}\right).$$

$$900 \quad \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right).$$

901 Possible choice of the two vectors is

$$\frac{1}{\sqrt{26}}(0, -4, 3, 1), \quad \frac{1}{3\sqrt{26}}(-13, 5, 6, 2).$$

$$902 \quad (1, 2, 1, 3), \quad (10, -1, 1, -3), \quad (19, -87, -61, 72).$$

$$903 \quad \text{For example, } \begin{pmatrix} 0, & 7, & 3, & -4 & -2 \\ 39, & -37, & 51, & -29 & 5 \end{pmatrix}.$$

904 The coefficients of the various equations determine  $m$  vectors. The system can be solved by finding a vector (or all the vectors) that are orthogonal to each of the given vectors. The given vectors span a subspace; the solutions fill out a space orthogonal to this subspace (its orthogonal complement). A basis for the solution set is a basis for this orthogonal complement.

$$905 \quad \text{For example, } \frac{1}{\sqrt{6}}(1, 0, 2, -1), \quad \frac{1}{\sqrt{498}}(1, 12, 8, 17).$$

$$906 \quad \begin{array}{ll} \text{a) } X' = (3, 1, -1, -2) \in P, & \text{b) } X' = (1, 7, 3, 3) \in P, \\ X'' = (2, 1, -1, 4) \perp P; & X'' = (-4, -2, 6, 0) \perp P. \end{array}$$



These equations are to be interpreted as follows. Each vector is given as a linear combination of the vectors in the first row of the determinant, the coefficients in this linear combination being the respective algebraic co-factors.

The following relations conclude the exercise:

$$Y^2 = Y(X - Z) = YX = \frac{1}{\Delta} \det \begin{bmatrix} 0 & -XA_1 & -XA_2 & \dots & -XA_m \\ A_1X & A_1^2 & A_1A_2 & \dots & A_1A_m \\ A_2X & A_2A_1 & A_2^2 & \dots & A_2A_m \\ \dots & \dots & \dots & \dots & \dots \\ A_mX & A_mA_1 & A_mA_2 & \dots & A_m^2 \end{bmatrix}$$

and

$$Z^2 = Z(X - Y) = ZX = \frac{1}{\Delta} \det \begin{bmatrix} X^2 & XA_1 & XA_2 & \dots & XA_m \\ A_1X & A_1^2 & A_1A_2 & \dots & A_1X_m \\ A_2X & A_2A_1 & A_2^2 & \dots & A_2A_m \\ \dots & \dots & \dots & \dots & \dots \\ A_mX & A_mA_1 & A_mA_2 & \dots & A_m^2 \end{bmatrix}.$$

908 Let  $Y$  be some vector in the space  $P$ , and let  $X'$  be the orthogonal projection of  $X$  onto  $P$ . Then

$$\begin{aligned} \cos(X, Y) &= \frac{XY}{|X| \cdot |Y|} = \frac{X'Y}{|X| \cdot |Y|} = \frac{|X'| \cdot |Y| \cdot \cos(X', Y)}{|X| \cdot |Y|} \\ &= \frac{|X'|}{|X|} \cos(X', Y), \end{aligned}$$

where  $|X|$  means the length of the vector  $X$ . This

formula shows that  $\cos(X, Y)$  is a maximum when  $\cos(X', Y) = 1$ , i.e.  $Y = \alpha X'$ ,  $\alpha > 0$ .

909 a)  $45^\circ$ ; b)  $90^\circ$ .

910  $\sqrt{\frac{m}{n}}.$

911  $|X - Y|^2 = |(X - X') + (X' - Y)|^2 = |X - X'|^2 + |X' - Y|^2$   
 $\geq |X - X'|^2$ , so that the equality can occur only for  $Y = X'$ .

912 a)  $\sqrt{7}$ ; b)  $\sqrt{\frac{2}{3}}.$

913  $\frac{n!}{(n+1)(n+2) \dots 2n \sqrt{2n+1}}.$

914 The shortest distance from  $X_0 + P$  to  $Y_0 + Q$  is equal to the shortest distance from the point  $X_0 - Y_0$  to the space  $P + Q$ .

915 Place the axis so that one of the vertices lies at the origin and suppose  $X_1, X_2, \dots, X_n$  are vectors from the origin to the remaining vertices. It is easy to see that  $X_i^2 = 1$ ,  $X_i X_j = \frac{1}{2}$ . The manifold spanned by the first  $m + 1$  vertices is the space  $t_1 X_1 + \dots + t_m X_m$ . The manifold spanned by the remaining  $n - m$  vertices is

$X_n + t_{m+1}(X_{m+1} - X_n) + \dots + t_{n-1}(X_{n-1} - X_n)$ . The shortest distance required is the distance from  $X_n$  to the space  $P$ , spanned by the vectors  $X_1, X_2, \dots, X_m, X_n - X_{m+1}, \dots, X_n - X_{n-1}$ .

Set  $X_n =$

$$t_1 X_1 + \dots + t_m X_m + t_{m+1} (X_n - X_{m+1}) + \dots + t_{n-1} (X_n - X_{n-1}) + Y,$$

where  $Y \perp P$ . We can obtain simultaneous equations to solve for  $t_1, \dots, t_{n-1}$  by forming the inner product of  $X_n$  with  $X_n, X_1, \dots, X_m, X_n - X_{m+1}, \dots, X_n - X_{n-1}$ :

$$\begin{aligned} t_1 + \frac{1}{2} t_2 + \dots + \frac{1}{2} t_m &= \frac{1}{2}, & t_{m+1} + \frac{1}{2} t_{m+2} + \dots + \frac{1}{2} t_{n-1} &= \frac{1}{2}, \\ \frac{1}{2} t_1 + t_2 + \dots + \frac{1}{2} t_m &= \frac{1}{2}, & \frac{1}{2} t_{m+1} + t_{m+2} + \dots + \frac{1}{2} t_{n-1} &= \frac{1}{2}, \\ \dots & \dots & \dots & \dots \\ \frac{1}{2} t_1 + \frac{1}{2} t_2 + \dots + t_m &= \frac{1}{2}, & \frac{1}{2} t_{m+1} + \frac{1}{2} t_{m+2} + \dots + t_{n-1} &= \frac{1}{2}, \end{aligned}$$

$$\text{Thus } t_1 = t_2 = \dots = t_m = \frac{1}{m+1}, \quad t_{m+1} = t_{m+2} = \dots = t_{n-1} = \frac{1}{n-m}.$$

$$\text{Therefore } Y = \frac{X_{m+1} + X_{m+2} + \dots + X_n}{n-m} - \frac{X_1 + X_2 + \dots + X_m}{m+1}.$$

Thus the vector that is the common perpendicular to the spaces in question runs between the centroids of their boundary points. The shortest distance itself is given by the length of this vector:

$$|Y| = \sqrt{\frac{(n+1)}{2(n-m)(m+1)}}.$$

916 a) The projection of the vector

$(t_1 + 2t_2, t_1 - 2t_2, t_1 + 5t_2, t_1 + 2t_2)$  on the first plane is  $(t_1 + 2t_2, t_1 - 2t_2, 0, 0)$ . Thus

$$\cos^2 \varphi = \frac{2t_1^2 + 8t_2^2}{4t_1^2 + 14t_1 t_2 + 37t_2^2} = \frac{2\lambda^2 + 8}{4\lambda^2 + 14\lambda + 37}, \quad \text{where } \lambda = \frac{t_1}{t_2}.$$

The value of this fraction is a maximum, namely  $8/9$ , for  $\lambda = -4$ .



b) The angle between an arbitrary vector of the second space and its orthogonal projection on the first space is constant and equal to  $\pi/4$ .

917 A cube is the set of points the coordinates of which satisfy the inequalities  $-\frac{a}{2} \leq x_i \leq \frac{a}{2}$ ,  $i=1, 2, 3, 4$ . Here  $a$  is the length of an edge. Introduce new axes with basis

$$e'_1 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \quad e'_2 = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right),$$

$$e'_3 = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right), \quad e'_4 = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right).$$

These vectors are orthogonal; they all have the same length; and their directions coincide with the directions of the diagonals of the cube. In terms of the new axes, the points interior to the cube satisfy the inequalities

$$-a \leq x'_1 + x'_2 + x'_3 + x'_4 \leq a, \quad -a \leq x'_1 + x'_2 - x'_3 - x'_4 \leq a,$$

$$-a \leq x'_1 - x'_2 + x'_3 - x'_4 \leq a, \quad -a \leq x'_1 - x'_2 - x'_3 + x'_4 \leq a.$$

We can obtain the intersection asked for by setting  $x'_1 = 0$ . It is the solid that lies in the space spanned by  $e'_2, e'_3, e'_4$ ; the coordinates of the points in this solid satisfy the inequalities  $\pm x'_2 \pm x'_3 \pm x'_4 \leq a$ .

This is a regular octohedron bounded by planes that have intercept  $a$  on the coordinate axes.

$$918 \quad V^2[B_1, B_2, \dots, B_m] = \det \begin{bmatrix} B_1^2 & B_1 B_2 & \dots & B_1 B_m \\ B_2 B_1 & B_2^2 & \dots & B_2 B_m \\ \dots & \dots & \dots & \dots \\ B_m B_1 & B_m B_2 & \dots & B_m^2 \end{bmatrix} = \det[B'B] = \{\det B\}^2.$$

It is easy to establish this formula by induction, using the result of problem 907. The nature of the formula shows that the volume has a value independent of the way the vertices are numbered. It is also clear that

$$V[cB_1, B_2, \dots, B_m] = |c| \cdot V[B_1, B_2, \dots, B_m].$$

Now let  $B_1 = B'_1 + B''_1$ ; let the orthogonal projections of  $B_1, B'_1, B''_1$  on the orthogonal complement of the space  $(B_2, \dots, B_m)$  be  $C_1, C'_1, C''_1$ .

It is clear that  $C_1 = C'_1 + C''_1$ . By definition,  $V[B_1, B_2, \dots, B_m] = |C_1| \cdot V[B_2, \dots, B_m]$ ,  $V[B'_1, B_2, \dots, B_m] = |C'_1| \cdot V[B_2, \dots, B_m]$ ,  $V[B''_1, B_2, \dots, B_m] = |C''_1| \cdot V[B_2, \dots, B_m]$ . Since  $|C_1| \leq |C'_1| + |C''_1|$ , it follows that

$$V[B_1, B_2, \dots, B_m] \leq V[B'_1, B_2, \dots, B_m] + V[B''_1, B_2, \dots, B_m].$$

Indeed equality is only possible if  $C'_1, C''_1$  are collinear and point in the same direction. But this can happen if and only if  $B'_1, B''_1$  lie in the space spanned by  $B_1, B_2, \dots, B_m$ , and even then only if in the formulas for expressing  $B'_1, B''_1$  in terms of  $B_1, B_2, \dots, B_m$  the two coefficients of  $B_1$  have the same sign. This last requirement is paraphrased geometrically by asserting that  $B'_1, B''_1$  lie on the same side of the space  $(B_2, \dots, B_m)$  in the space  $(B_1, B_2, \dots, B_m)$ .

$$919 \quad V^2[B_1, B_2, \dots, B_n] = \det \begin{bmatrix} B_1^2 & B_1 B_2 & \dots & B_1 B_n \\ B_2 B_1 & B_2^2 & \dots & B_2 B_n \\ \dots & \dots & \dots & \dots \\ B_n B_1 & B_n B_2 & \dots & B_n^2 \end{bmatrix} = \det[B'B] = \{\det B\}^2,$$

where  $B$  is the matrix, the rows of which are the coordinates of the vectors  $B_1, B_2, \dots, B_n$ .

920 The following two properties of "volume" follow directly from the definition:

$$d) \quad V[B_1 + X, B_2, \dots, B_m] = V[B_1, B_2, \dots, B_m]$$

where  $X$  is an arbitrary vector lying in the space  $(B_2, \dots, B_m)$ , and the points  $B_1, B_1 + X$  are at the same distance from  $(B_2, \dots, B_m)$ .

$$e) \quad V[B_1, B_2, \dots, B_m] \leq |B_1| \cdot V[B_2, \dots, B_m].$$

This follows from the fact that the "altitude," that is, the length of the component of the vector  $B_1$  that is orthogonal to  $(B_2, \dots, B_m)$  is no greater than the length of the vector  $B_1$  itself.

Now let  $C_1, C_2, \dots, C_m$  be the orthogonal projections of the vector  $B_1, B_2, \dots, B_m$  onto a space  $P$ . Suppose as an induction hypothesis that the relation  $V[C_2, \dots, C_m] \leq V[B_2, \dots, B_m]$  has already been proved. Let  $B'_1$  denote the component of the vector  $B_1$  that is orthogonal to  $(B_2, \dots, B_m)$ . Let  $C'_1$  be the projection of  $B_1$  on  $P$ . Since  $B'_1 - B_1 \in (B_2, \dots, B_m)$ , we conclude that  $C'_1 - C \in (C_2, \dots, C_m)$ . Therefore

$$V[C_1, C_2, \dots, C_m] = V[C'_1, C_2, \dots, C_m] \leq |C'_1| \cdot V[C_2, \dots, C_m].$$

Since the relation  $|C'_1| \leq |B'_1|$  is true, the induction hypothesis  $V[C_2, \dots, C_m] \leq V[B_2, \dots, B_m]$

gives the result

$$V[C_1, C_2, \dots, C_m] \leq |B'_1| \cdot V[B_2, \dots, B_m] = V[B_1, B_2, \dots, B_m].$$

To complete the proof one need merely remark the truth of the result in one dimension.

- 921 If every vector  $A_i$  is orthogonal to every vector  $B_j$ , the formula

$$V[A_1, \dots, A_m, B_1, \dots, B_k] = V[A_1, \dots, A_m] \cdot V[B_1, \dots, B_k]$$

follows from the formula for the square of a general volume. To establish the theorem in general we define  $C_1, \dots, C_k$  as the projections of  $B_1, \dots, B_k$  on the orthogonal complement of  $(A_1, \dots, A_m)$ . The preceding problem gives the result  $V[C_1, \dots, C_k] \leq V[B_1, \dots, B_k]$ , and the following chain of relations establishes the general result.

$$\begin{aligned} V[A_1, \dots, A_m, B_1, \dots, B_k] &= V[A_1, \dots, A_m, C_1, \dots, C_k] = \\ &= V[A_1, \dots, A_m] \cdot V[C_1, \dots, C_k] \leq V[A_1, \dots, A_m] \cdot V[B_1, \dots, B_k]. \end{aligned}$$

See problem 518, which is essentially equivalent to this one.

- 922 As shown in 920,  $V[A_1, \dots, A_m] \leq |A_1| \cdot V[A_2, \dots, A_m]$ . The result follows directly from this formula.

See problem 519, which is essentially identical with this one.

923 Similar solids in  $n$ -dimensional space have volumes proportional to the  $n$ -th power of the coefficient of similarity. This is clear for parallelepipeds, and the volume of a general solid can be thought of as the limit of the sum of volumes of parallelepipeds. Therefore the volume  $V_n(R)$  of an  $n$ -dimensional sphere of radius  $R$  is  $V_n(1)R^n$ .

The volume of  $V_n(1)$  can be computed by the method of Archimedes. We cut the sphere by  $(n-1)$ -dimensional hyperplanes and use Cavalieri's principle.

Let  $x$  be the distance of a typical hyperplane from the center. The trace of this hyperplane is a  $(n-1)$ -dimensional sphere of radius  $\sqrt{1-x^2}$ . Therefore

$$\begin{aligned} V_n(1) &= 2 \int_0^1 V_{n-1}(\sqrt{1-x^2}) dx = 2V_{n-1}(1) \int_0^1 (1-x^2)^{\frac{n-1}{2}} dx \\ &= V_{n-1}(1) \int_0^1 t^{\frac{n-1}{2}} (1-t)^{-\frac{1}{2}} dt = V_{n-1}(1) B\left(\frac{n+1}{2}, \frac{1}{2}\right) \\ &= V_{n-1}(1) \cdot \frac{\Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}. \end{aligned}$$

From this, it follows that  $V_n(1) = \frac{n^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$ .

924 The polynomials  $1, x, \dots, x^n$  form a basis. The square of the volume of the corresponding parallelepipeds is

$$\det \begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n+1} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+2} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n+1} \end{bmatrix} = \frac{[1!2! \dots n!]^3}{(n+1)!(n+2)! \dots (2n+1)!}.$$

925

- a)  $\lambda_1 = 1, X_1 = c(1, -1); \lambda_2 = 3, X_2 = c(1, 1);$   
 b)  $\lambda_1 = 7, X_1 = c(1, 1); \lambda_2 = -2, X_2 = c(4, -5);$   
 c)  $\lambda_1 = at, X_1 = c(1, t); \lambda_2 = -at, X_2 = c(1, -t);$   
 d)  $\lambda_1 = 2, X_1 = c_1(1, 1, 0, 0) + c_2(1, 0, 1, 0) + c_3(1, 0, 0, 1);$   
 $\lambda_2 = -2, X_2 = c(1, -1, -1, -1);$   
 e)  $\lambda = 2, X = c_1(-2, 1, 0) + c_2(1, 0, 1);$   
 f)  $\lambda = -1, X = c(1, 1, 1, -1);$   
 g)  $\lambda_1 = 1, X_1 = c_1(1, 0, 1) + c_2(0, 1, 0); \lambda_2 = -1, X_2 = c(1, 0, -1);$   
 h)  $\lambda_1 = 0, X_1 = c(3, -1, 2); \lambda_{2,3} = \pm \sqrt{-14},$   
 $X_{2,3} = c(3 \pm 2\sqrt{-14}, 13, 2 \mp 3\sqrt{-14});$   
 i)  $\lambda_1 = 1, X_1 = c(3, -6, 20); \lambda_2 = -2, X_2 = c(0, 0, 1);$   
 j)  $\lambda_1 = 1, X_1 = c(1, 1, 1); \lambda_2 = \epsilon, X_2 = c(3 + 2\epsilon, 2 + 3\epsilon, 3 + 3\epsilon);$   
 $\lambda_3 = \epsilon^2, X_3 = c(3 + 2\epsilon^2, 2 + 3\epsilon^2, 3 + 3\epsilon^2),$

where  $\epsilon = -\frac{1}{2} + \frac{i\sqrt{3}}{2}.$

926

The proper values of the matrix  $A^{-1}$  are the reciprocals of the proper values of the matrix  $A$ . For if  $\det [A^{-1} - \lambda E] = 0$ , then  $\det [E - \lambda A] = 0, \det [A - \lambda^{-1} E] = 0.$

- 927 The proper values of the matrix  $A^2$  are the squares of the proper values of the matrix  $A$ . Indeed suppose  $\det[A - \lambda E] = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_n - \lambda)$ . Then  $\det[A + \lambda E] = (\lambda_1 + \lambda)(\lambda_2 + \lambda) \dots (\lambda_n + \lambda)$ . If these two equations are multiplied, and if  $\lambda^2$  is replaced by  $\mu$ , we obtain

$$\det[A^2 - \mu E] = (\lambda_1^2 - \mu)(\lambda_2^2 - \mu) \dots (\lambda_n^2 - \mu).$$

If the proper values of a matrix are defined in the intrinsic way:  $A\chi = \chi\lambda$ , then it follows that  $A^2\chi = A\chi\lambda = \chi\lambda^2$ . This argument must be slightly modified if the matrix  $A$  is deficient, that is if it has fewer than  $n$  proper vectors. This modification can consist either of a continuity argument, or a full use of the Jordan normal form.

- 928 The proper values of the matrix  $A^m$  are the  $m$ -th powers of the proper values of the matrix  $A$ .

The proof is similar to that of problem 927; the relation

$$\det[A - \lambda E] = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_n - \lambda)$$

is changed to  $n - 1$  other relations by replacing  $\lambda$  by  $\lambda\epsilon$ ,  $\lambda\epsilon^2$ ,  $\dots$ ,  $\lambda\epsilon^{n-1}$ , where  $\epsilon$  is a primitive  $n$ -th root of unity. As in the preceding problem the  $n$  equations are then multiplied. Finally  $\lambda^n$  is replaced by  $\mu$ . The alternative argument of problem 927 can also be used.

929 From the relation  $f(A) = b_0(A - \xi_1 E) \dots (A - \xi_m E)$ ,  
it follows that  $\det[F(A)] = b_0^n \cdot \det[A - \xi_1 E] \dots$   
 $\det[A - \xi_m E] = b_0^n F(\xi_1) \dots F(\xi_m)$ .

930 Let  $F(\lambda) = |A - \lambda E| = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_n - \lambda)$ ,  
and  $f(x) = b_0(x - \xi_1)(x - \xi_2) \dots (x - \xi_m)$ . Then

$$\det[f(A)] = b_0^n \prod_{i=1}^n \prod_{k=1}^m (\lambda_i - \xi_k) = f(\lambda_1) f(\lambda_2) \dots f(\lambda_n).$$

931 Set  $\varphi(x) = f(x) - \lambda$  and apply the result of the  
preceding problem to obtain:

$$\det[f(A) - \lambda E] = (f(\lambda_1) - \lambda)(f(\lambda_2) - \lambda) \dots (f(\lambda_n) - \lambda).$$

This shows that the proper values of the matrix  
 $f(A)$  are  $f(\lambda_1), f(\lambda_2), \dots, f(\lambda_n)$ . The second proof  
in 927 can be applied if  $f$  is any function  
expansible in a convergent infinite series.

The proper values of a triangular matrix are  
the diagonal elements. The proper values of any  
matrix are the same as those of a similar matrix.  
But by Schur's theorem every matrix is similar to  
a triangular matrix. A very short argument shows  
that the diagonal elements of a power of a triangular  
matrix are the corresponding powers of the diagonal  
elements of the original matrix. Thus we have  
another proof of the results of problems 927, 928,  
929, 930, and 931.

Schur's theorem is easily established by  
induction; the induction hypothesis is that the  
theorem is valid for every matrix of dimension  
 $n - 1$ . To complete the proof it is only necessary



to know that every  $n$ -th order matrix has at least one proper value. Let  $\lambda$  be a proper value of the matrix  $A$ ; let  $e_1$  be the corresponding proper vector. Construct the matrix  $U$ , a unitary matrix having  $e_1$  for first row (the possibility of constructing such a matrix is known as the Gram-Schmidt orthogonalization process). Then  $U$  transforms  $A$  into block triangular form:

$$U^*AU = \begin{bmatrix} \lambda & a \\ 0 & A_1 \end{bmatrix}.$$

By the induction hypothesis, there is a unitary matrix

$$\text{diag}\{1, V\}$$

that transforms  $U^*AU$  into triangular form. Therefore  $U \cdot \text{diag}\{1, V\}$  transforms  $A$  into triangular form, as was to be proved.

932 Let  $X$  be a proper vector of the matrix  $A$ , and let the corresponding proper value be  $\lambda$ . Then

$$\begin{aligned} EX &= X, \\ AX &= X\lambda, \\ A^2X &= X\lambda^2 \\ &\dots\dots\dots \\ A^mX &= X\lambda^m \end{aligned}$$

If the vector equations in this array are multiplied by arbitrary scalars, and added, we obtain the following relation for an arbitrary polynomial  $f$ :

$$f(A) X = X f(\lambda).$$

Thus  $X$  is a proper vector of  $f(A)$  and the corresponding proper value is  $f(\lambda)$ .

If  $A$  is not deficient, that is if  $A$  has  $n$  linearly independent proper vectors, then, taking  $f$  as the characteristic polynomial of  $A$ , we see that  $f(A) X = 0$  for every proper vector. But this relation must then be true for every vector in the space, since the proper vectors are linearly independent, and therefore span the space. In this case therefore, the polynomial  $f(A)$  must be the null matrix, since this is the only matrix that transforms the entire space into the null space. Thus in this case it has been proved that the matrix  $A$  satisfies its characteristic equation.

- 933 The proper values of  $A^2$  are  $n, -n$ , with multiplicity  $(n + 1)/2, (n - 1)/2$ . Thus the proper values of  $A$  are  $+\sqrt{n}, -\sqrt{n}, +i\sqrt{n}, -i\sqrt{n}$ . If the multiplicities of these roots are  $a, b, c, d$ , we have  $a + b = (n + 1)/2, c + d = (n - 1)/2$ . The sum of the proper values of a matrix is the sum of the elements in the principal diagonal. Thus

$$[a - b + (c - d)i] \sqrt{n} = 1 + \varepsilon + \varepsilon^4 + \dots + \varepsilon^{(n-1)^4}.$$

By problem 126, the absolute value of the right member of this equation is  $\sqrt{n}$ . Therefore

$$(a - b)^2 + (c - d)^2 = 1.$$

Note that  $c - d, c + d$  are both even or both odd;

thus

$$a - b = 0, \quad c - d = \pm 1, \quad \text{if } (n-1)/2 \text{ is odd}$$

and

$$a - b = \pm 1, \quad c - d = 0, \quad \text{if } (n-1)/2 \text{ is even.}$$

Finally we obtain for  $n = 1 + 4k$ :

$$c = d = k; \quad a = k + 1, \quad b = k \quad \text{or} \quad a = k, \quad b = k + 1;$$

for  $n = 3 + 4k$

$$a = b = k + 1; \quad c = k + 1, \quad d = k \quad \text{or} \quad c = k, \quad d = k + 1$$

At the moment, proper values are determined to within sign. To obtain the sign we use the fact that the product of the proper values is the determinant of the matrix. Problem 299 enables us to show that for  $n = 1 + 4k$ , we have

$$a = k + 1, \quad b = k;$$

for  $n = 3 + 4k$ , we have

$$c = k + 1, \quad d = k.$$

This determines the characteristic numbers completely.

$$\begin{aligned} 934 \quad 1 + \varepsilon + \varepsilon^4 + \dots + \varepsilon^{(n-1)^2} &= +\sqrt[n]{n} \quad \text{for } n = 4k + 1, \\ 1 + \varepsilon + \varepsilon^4 + \dots + \varepsilon^{(n-1)^2} &= +i\sqrt[n]{n} \quad \text{for } n = 4k + 3. \end{aligned}$$

$$935 \quad \text{Set } \frac{x}{y} = \alpha^n. \quad \text{Then } \lambda_k = y \frac{\alpha \varepsilon_k - \alpha^n}{1 - \alpha \varepsilon_k}, \quad \text{where}$$

$$\varepsilon_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}, \quad k = 0, 1, \dots, n-1.$$

$$b) \quad \lambda_k = a_1 + a_2 \varepsilon_k + a_3 \varepsilon_k^2 + \dots + a_n \varepsilon_k^{n-1}.$$

$$c) \quad \lambda_k = 2i \cos \frac{k\pi}{n+1}, \quad k = 1, 2, \dots, n.$$

936

$$A \times B - \lambda E_{mn} = \begin{pmatrix} a_{11}B - \lambda E_m & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B - \lambda E_m & \dots & a_{2n}B \\ \dots & \dots & \dots & \dots \\ a_{n1}B & a_{n2}B & \dots & a_{nn}B - \lambda E_m \end{pmatrix},$$

so that by problem 537, we see that

$$\det[A \times B - \lambda E_{mn}] = \det[\varphi(B)],$$

where

$$\varphi(x) = |Ax - \lambda E_n| = \prod_{i=1}^n (\alpha_i x - \lambda).$$

By problem 930,

$$\det[\varphi(B)] = \prod_{k=1}^m \varphi(\beta_k) = \prod_{i=1}^n \prod_{k=1}^m (\alpha_i \beta_k - \lambda).$$

Thus the proper values of  $A \times B$  are the numbers  $\alpha_i \beta_k$ , where  $\alpha_i$  are the proper values of  $A$ ,  $\beta_k$  are the proper values of  $B$ .

937 If  $A$  is a nonsingular matrix, then

$$\begin{aligned} \det[BA - \lambda E] &= \det[A^{-1}(AB - \lambda E)A] = \\ \det[A^{-1}] \cdot \det[AB - \lambda E] \cdot \det[A] &= \det[AB - \lambda E]. \end{aligned}$$

To improve this result to the case of singular matrices, we can think of it as an identity in the  $n^2$  elements of  $A$ , since it has been proved in all but a minority of the cases. Indeed  $A$  is singular only when its coefficients satisfy a polynomial equation. A more direct proof consists in noting the relations

$$\det[B(A - \mu I) - \lambda I] = \det[A - \mu I] \det[B - \lambda I],$$

which is obviously true for all but a finite number of values of  $\mu$ . Since this is a polynomial

identity, it must be true for all the values of  $\mu$ , including  $\mu = 0$ .

- 938 Complete the matrices  $A, B$  to square matrices  $A', B'$  of order  $n$  by appending  $n - m$  rows of zeros to  $A$ , and  $n - m$  columns of zeros to  $B$ . Then  $BA = B'A'$ , and  $A'B'$  can be obtained from  $AB$  by adding a border of zeros. The result to be proved can be obtained by applying the preceding problem.

Problems 939, 940, 941 do not have unique answers. We give below the answers that are as close to triangular in appearance as possible.

- 939
- |  |                                  |
|--|----------------------------------|
| a) $x_1'^2 + x_2'^2 + x_3'^2$ ,                  | b) $-x_1'^2 + x_2'^2 + x_3'^2$ , |
| $x_1' = x_1 + x_2$ ,                             | $x_1' = x_1$ ,                   |
| $x_2' = x_2 + 2x_3$ ,                            | $x_2' = x_1 - 2x_2$ ,            |
| $x_3' = x_3$ ;                                   | $x_3' = x_1 + x_3$ ;             |
| c) $x_1'^2 - x_2'^2 - x_3'^2$ ,                  | d) $x_1'^2 + x_2'^2 - x_3'^2$ ,  |
| $x_1' = \frac{1}{2}x_1 + \frac{1}{2}x_2 + x_3$ , | $x_1' = x_1 - x_2 + x_3 - x_4$ , |
| $x_2' = \frac{1}{2}x_1 - \frac{1}{2}x_2$ ,       | $x_2' = x_2 + x_3 + x_4$ ,       |
| $x_3' = x_3$ ;                                   | $x_3' = x_2 - x_3 + 2x_4$ ,      |
|  | $x_4' = x_4$ ;                   |
| e) $x_1'^2 - x_2'^2 + x_3'^2 - x_4'^2$ ,         |                                  |
| $x_1' = x_1 + \frac{1}{2}x_2$ ,                  |                                  |
| $x_2' = \frac{1}{2}x_2$ ,                        |                                  |
| $x_3' = \frac{1}{2}x_3 + \frac{1}{2}x_4$ ,       |                                  |
| $x_4' = \frac{1}{2}x_3 - \frac{1}{2}x_4$ .       |                                  |

940 
$$x_1'^2 + \frac{3}{4} x_2'^2 + \frac{4}{6} x_3'^2 + \dots + \frac{n+1}{2n} x_n'^2.$$

Here we give just the matrix of coefficients for expressing  $x'_1, x'_2, \dots, x'_n$  in terms of  $x_1, x_2, \dots, x_n$ :

$$\begin{pmatrix} 1 & 1/2 & 1/2 & \dots & 1/2 \\ 0 & 1 & 1/3 & \dots & 1/3 \\ 0 & 0 & 1 & 1/4 & \dots & 1/4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

$$\begin{aligned} 941 \quad & \left( \frac{x_1 + x_2}{2} + x_3 + x_4 + \dots + x_n \right)^2 - \left( \frac{x_1 - x_2}{2} \right)^2 \\ & - \left( x_3 + \frac{1}{2} x_4 + \dots + \frac{1}{2} x_n \right)^2 \\ & - \frac{3}{4} \left( x_4 + \frac{1}{3} x_5 + \dots + \frac{1}{3} x_n \right)^2 - \dots - \frac{n-1}{2(n-2)} x_n^2. \end{aligned}$$

942 The matrix of a positive definite quadratic form  
has the form  $A'A$ , where  $A$  is the real nonsingular  
matrix of coefficients needed to transform the  
form into the sum of squares of new variables.

To show that the minors are positive, use the result of problem 510.

[illegible]

be a quadratic form. Set

$$f^{(k)} = a_{11}x_1^2 + \dots + a_{1k}x_1x_k + \dots + a_{k1}x_kx_1 + \dots + a_{kk}x_k^2.$$

$$\Delta_k = \det \begin{bmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{kk} \end{bmatrix} ; \text{ Let } r \text{ be the rank of the form } f.$$



is given by  $\alpha_{k+1}\alpha_{k+2}\dots\alpha_n = \Delta_n/\Delta_k$ .

944 In problem 942 Sylvester's condition was shown to be necessary. In problem 943 it was shown to be sufficient.

945 Let  $\ell$  be some linear form in the variables  $x_1, x_2, \dots, x_n$ . We transform the form  $f$  by a sequence of transformations all of which have unit determinant, and take the form  $\ell$  as the last of the new variables. Finally we transform  $f$  into canonical form by performing a triangular transformation. The form  $f$  takes the form

$$f = \alpha_1 x_1'^2 + \alpha_2 x_2'^2 + \dots + \alpha_n x_n'^2,$$

where  $x_n' = \ell$ .

The discriminant of the form  $f$  is  $\alpha_1\alpha_2\dots\alpha_n$ .

The discriminant of the form  $f + \ell^2$  is  $\alpha_1\alpha_2\dots\alpha_{n-1}(\alpha_n + 1)$ . It exceeds the discriminant of the form  $f$ , since all the coefficients  $\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \alpha_n$  are positive.

946

$$\begin{aligned} f(x_1, x_2, \dots, x_n) &= a_{11}x_1^2 + 2a_{21}x_1x_2 + \dots + 2a_{n1}x_1x_n + \varphi \\ &= a_{11}\left(x_1 + \frac{a_{21}}{a_{11}}x_2 + \dots + \frac{a_{n1}}{a_{11}}x_n\right)^2 + f_1(x_2, \dots, x_n), \end{aligned}$$

where  $f_1 = \varphi - a_{11}\left(\frac{a_{21}}{a_{11}}x_2 + \dots + \frac{a_{n1}}{a_{11}}x_n\right)^2$ .

The form  $f_1$  is positive and its discriminant is  $D_f/a_{11}$ , where  $D_f$  is the discriminant of  $f$ . By problem 945,  $D_{f_1} \geq \frac{D_f}{a_{11}}$ , and the assertion is proved.



947 The proof is similar to that of Sylvester's law of inertia.

948 We construct the linear form

$$l_k = u_1 + u_2 x_k + \dots + u_n x_k^{n-1}, \quad k = 1, 2, \dots, n,$$

where  $x_1, x_2, \dots, x_n$  are roots of the given equation.

In this construction equal roots will correspond to equal forms, different roots to linearly independent ones, real roots to real forms, conjugate complex roots to conjugate complex forms.

The real and imaginary parts of a complex form  $l_k = \lambda_k + i\mu_k$  are linearly independent of each other and of any forms that correspond to roots distinct from  $x_k, \bar{x}_k$ .

If we construct the quadratic form

$$f(u_1, \dots, u_n) = \sum_{k=1}^n (u_1 + u_2 x_k + \dots + u_n x_k^{n-1})^2$$

we have a form the rank of which is the number of distinct roots of the given equation. Its matrix of coefficients is

$$\begin{pmatrix} s_0 & s_1 & \dots & s_{n-1} \\ s_1 & s_2 & \dots & s_n \\ \cdot & \cdot & \cdot & \cdot \\ s_{n-1} & s_n & \dots & s_{2n-2} \end{pmatrix}.$$

The sum of the squares of the complex conjugate linear forms  $l_k = \lambda_k + i\mu_k$ ,  $\bar{l}_k = \lambda_k - i\mu_k$  is  $2\lambda_k^2 - 2\mu_k^2$ . Therefore by the law of inertia, the number of negative terms in any canonical decomposition of  $f$  is equal to the number of

distinct pairs of conjugate complex roots of the given equation.

949 Follows from 948, 944.

950 The definition  $(f, \varphi)$  is obviously distributive. Thus it is sufficient to prove the assertion for squares of linear forms.

Let

$$f = (\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n)^2,$$

$$\varphi = (\beta_1 x_1 + \beta_2 x_2 + \dots + \beta_n x_n)^2.$$

It is easy to see that

$$(f, \varphi) = (\alpha_1 \beta_1 x + \alpha_2 \beta_2 x_2 + \dots + \alpha_n \beta_n x_n)^2 \geq 0.$$

- 951 a)  $4x_1'^2 + x_2'^2 - 2x_3'^2$ ,  $x_1' = \frac{2}{3}x_1 - \frac{2}{3}x_2 + \frac{1}{3}x_3$ ,  
 $x_2' = \frac{2}{3}x_1 + \frac{1}{3}x_2 - \frac{2}{3}x_3$ ,  
 $x_3' = \frac{1}{3}x_1 + \frac{2}{3}x_2 + \frac{2}{3}x_3$ ;  
 b)  $2x_1'^2 - x_2'^2 + 5x_3'^2$ ,  $x_1' = \frac{2}{3}x_1 - \frac{1}{3}x_2 - \frac{2}{3}x_3$ ,  
 $x_2' = \frac{2}{3}x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_3$ ,  
 $x_3' = \frac{1}{3}x_1 - \frac{2}{3}x_2 + \frac{2}{3}x_3$ ;  
 c)  $7x_1'^2 + 4x_2'^2 + x_3'^2$ ,  $x_1' = \frac{1}{3}x_1 + \frac{2}{3}x_2 - \frac{2}{3}x_3$ ,  
 $x_2' = \frac{2}{3}x_1 + x_2 + \frac{2}{3}x_3$ ,  
 $x_3' = -\frac{2}{3}x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_3$ ;  
 d)  $10x_1'^2 + x_2'^2 + x_3'^2$ ,  $x_1' = \frac{1}{3}x_1 + \frac{2}{3}x_2 - \frac{2}{3}x_3$ ,  
 $x_2' = \frac{2\sqrt{5}}{5}x_1 - \frac{\sqrt{5}}{5}x_2$ ,  
 $x_3' = \frac{2\sqrt{5}}{15}x_1 + \frac{4\sqrt{15}}{15}x_2 + \frac{\sqrt{5}}{3}x_3$ ;

- e)  $-7x_1'^2 + 2x_2'^2 + 2x_3'^2$ ,  $x_1' = \frac{1}{3}x_1 + \frac{2}{3}x_2 - \frac{2}{3}x_3$ ,  
 $x_2' = \frac{2\sqrt{5}}{5}x_1 - \frac{\sqrt{5}}{5}x_2$ ,  
 $x_3' = \frac{2\sqrt{5}}{15}x_1 + \frac{4\sqrt{5}}{15}x_2 + \frac{\sqrt{5}}{3}x_3$ ;
- f)  $2x_1'^2 + 5x_2'^2 + 8x_3'^2$ ,  $x_1' = \frac{2}{3}x_1 + \frac{1}{3}x_2 + \frac{2}{3}x_3$ ,  
 $x_2' = \frac{1}{3}x_1 + \frac{2}{3}x_2 - \frac{2}{3}x_3$ ,  
 $x_3' = \frac{2}{3}x_1 - \frac{2}{3}x_2 - \frac{1}{3}x_3$ ;
- g)  $7x_1'^2 - 2x_2'^2 + 7x_3'^2$ ,  $x_1' = \frac{\sqrt{2}}{2}x_1 - \frac{\sqrt{2}}{2}x_3$ ,  
 $x_2' = \frac{2}{3}x_1 + \frac{1}{3}x_2 + \frac{2}{3}x_3$ ,  
 $x_3' = \frac{\sqrt{2}}{6}x_1 - \frac{2\sqrt{2}}{3}x_2 + \frac{\sqrt{2}}{6}x_3$ .
- h)  $11x_1'^2 + 5x_2'^2 - x_3'^2$ ,  $x_1' = \frac{2}{3}x_1 - \frac{2}{3}x_2 - \frac{1}{3}x_3$ ,  
 $x_2' = \frac{2}{3}x_1 + x_2 + \frac{2}{3}x_3$ ,  
 $x_3' = \frac{1}{3}x_1 + \frac{2}{3}x_2 - \frac{2}{3}x_3$ ;
- i)  $x_1'^2 - x_2'^2 + 3x_3'^2 + 5x_4'^2$ ,  $x_1' = \frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4$ ,  
 $x_2' = \frac{1}{2}x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4$ ,  
 $x_3' = \frac{1}{2}x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_4$ ,  
 $x_4' = \frac{1}{2}x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{1}{2}x_4$ ;
- j)  $x_1'^2 + x_2'^2 - x_3'^2 - x_4'^2$ ,  $x_1' = \frac{\sqrt{2}}{2}x_1 + \frac{\sqrt{2}}{2}x_2$ ,  
 $x_2' = \frac{\sqrt{2}}{2}x_3 + \frac{\sqrt{2}}{2}x_4$ ,  
 $x_3' = \frac{\sqrt{2}}{2}x_1 - \frac{\sqrt{2}}{2}x_2$ ,  
 $x_4' = \frac{\sqrt{2}}{2}x_3 - \frac{\sqrt{2}}{2}x_4$ ;

$$\text{k) } x_1'^2 + x_2'^2 + 3x_3'^2 - x_4'^2, \quad x_1' = \frac{\sqrt{2}}{2} x_1 + \frac{\sqrt{2}}{2} x_4,$$

$$x_2' = \frac{\sqrt{2}}{2} x_1 + \frac{\sqrt{2}}{2} x_3,$$

$$x_3' = \frac{1}{2} x_1 + \frac{1}{2} x_2 - \frac{1}{2} x_3 - \frac{1}{2} x_4,$$

$$x_4' = -\frac{1}{2} x_1 + \frac{1}{2} x_2 + \frac{1}{2} x_3 - \frac{1}{2} x_4;$$

$$\text{l) } x_1'^2 + x_2'^2 + x_3'^2 - 3x_4'^2, \quad x_1' = \frac{\sqrt{2}}{2} x_1 + \frac{\sqrt{2}}{2} x_2,$$

$$x_2' = \frac{\sqrt{2}}{2} x_3 + \frac{\sqrt{2}}{2} x_4,$$

$$x_3' = \frac{1}{2} x_1 - \frac{1}{2} x_2 + \frac{1}{2} x_3 - \frac{1}{2} x_4,$$

$$x_4' = \frac{1}{2} x_1 - \frac{1}{2} x_2 - \frac{1}{2} x_3 + \frac{1}{2} x_4;$$

$$\text{m) } x_1'^2 - x_2'^2 + 7x_3'^2 - 3x_4'^2, \quad x_1' = \frac{1}{2} x_1 + \frac{1}{2} x_2 + \frac{1}{2} x_3 + \frac{1}{2} x_4,$$

$$x_2' = \frac{1}{2} x_1 + \frac{1}{2} x_2 - \frac{1}{2} x_3 - \frac{1}{2} x_4,$$

$$x_3' = \frac{1}{2} x_1 - \frac{1}{2} x_2 + \frac{1}{2} x_3 - \frac{1}{2} x_4,$$

$$x_4' = \frac{1}{2} x_1 - \frac{1}{2} x_2 - \frac{1}{2} x_3 + \frac{1}{2} x_4;$$

$$\text{n) } 5x_2'^2 - 5x_3'^2 + 3x_4'^2 - 3x_1'^2, \quad x_1' = \frac{1}{2} x_1 + \frac{1}{2} x_2 + \frac{1}{2} x_3 + \frac{1}{2} x_4,$$

$$x_2' = \frac{1}{2} x_1 + \frac{1}{2} x_2 - \frac{1}{2} x_3 - \frac{1}{2} x_4,$$

$$x_3' = \frac{1}{2} x_1 - \frac{1}{2} x_2 + \frac{1}{2} x_3 - \frac{1}{2} x_4,$$

$$x_4' = \frac{1}{2} x_1 - \frac{1}{2} x_2 - \frac{1}{2} x_3 + \frac{1}{2} x_4.$$

$$952 \quad \text{a) } \frac{n+1}{2} x_1'^2 + \frac{1}{2} (x_2'^2 + x_3'^2 + \dots + x_n'^2);$$

$$\text{b) } \frac{n-1}{2} x_1'^2 - \frac{1}{2} (x_2'^2 + x_3'^2 + \dots + x_n'^2),$$

where

$$x_1' = \frac{1}{\sqrt{n}} (x_1 + x_2 + \dots + x_n);$$

$$x_i' = \alpha_{i1} x_1 + \alpha_{i2} x_2 + \dots + \alpha_{in} x_n, \quad i = 2, \dots, n,$$

where  $(\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{in})$  is an arbitrary orthonormal basis for the solutions of the equation

$$x_1 + x_2 + \dots + x_n = 0.$$

$$953 \quad x_1'^2 \cos \frac{\pi}{n+1} + x_2'^2 \cos \frac{2\pi}{n+1} + \dots + x_n'^2 \cos \frac{n\pi}{n+1}.$$

954 If every proper value of the matrix  $A$  lies in the segment  $[a, b]$ , then every proper value of the matrix  $A - \lambda E$  is negative for  $\lambda > b$  and positive for  $\lambda < a$ . Therefore the quadratic form of matrix  $A - \lambda E$  is negative definite for  $\lambda > b$ , and positive definite for  $\lambda < a$ . Conversely if the quadratic form  $(A - \lambda E) X \cdot X$  is negative definite for  $\lambda > b$ , and positive definite for  $\lambda < a$ , then every proper value for the matrix  $A - \lambda E$  is positive for  $\lambda < a$  and negative for  $\lambda > b$ . Therefore for every proper value of the matrix  $A$  lies in the segment  $[a, b]$ .

955 If  $X$  is an arbitrary vector, the following inequalities are satisfied:

$$aX \cdot X \leq AX \cdot X \leq cX \cdot X, \quad bX \cdot X \leq BX \cdot X \leq dX \cdot X.$$

$$\text{Thus } (a+b)X \cdot X \leq (A+B)X \cdot X \leq (c+d)X \cdot X.$$

Therefore every proper value of the matrix  $A + B$  lies in the segment  $[a+c, b+d]$ .

956 a) Follows from the result of problem 937.

$$b) \quad |AX|^2 = AX \cdot AX = X \cdot A'AX \leq |X|^2 \cdot \|A\|^2.$$

Equality can occur only if  $X$  is a proper vector of the matrix  $A'A$  corresponding to the proper value  $\|A\|^2$ .

c) Let  $X$  be an arbitrary vector; then

$$|(A+B)X| \leq |AX| + |BX| \leq (\|A\| + \|B\|)|X|.$$

But for arbitrary  $X_0$ ,

$$|(A+B)X_0| = (\|A+B\|) \cdot |X_0|.$$

Therefore

$$\|A+B\| \leq \|A\| + \|B\|.$$

d) Let  $X$  be an arbitrary vector; then

$$|ABX| \leq \|A\| \cdot |BX| \leq \|A\| \cdot \|B\| \cdot |X|.$$

Now let  $X$  be the vector  $X_0$  for which the relation

$$\|AB\| \cdot |X_0| = |ABX_0| \quad \text{holds, then} \quad \|AB\| \leq \|A\| \cdot \|B\|.$$

e) Let  $\lambda = p + qi$  be a proper value of the matrix  $A$ ;  $X = Y + iZ$  be a proper vector that corresponds, then

$$AY = pY - qZ, \quad AZ = qY + pZ;$$

thus

$$|pY - qZ|^2 \leq \|A\|^2 |Y|^2, \quad |qY + pZ|^2 \leq \|A\|^2 |Z|^2.$$

Having these inequalities, we obtain

$$|\lambda|^2 (|Y|^2 + |Z|^2) = (p^2 + q^2) (|Y|^2 + |Z|^2) \leq \|A\|^2 (|Y|^2 + |Z|^2)$$

and therefore  $|\lambda| \leq \|A\|$ .

- 957 Let  $A$  be a real nonsingular matrix. Then  $A'AX \cdot X = |AX|^2$  is a positive quadratic form that can be transformed into canonical form  $B$  by a succession of triagonal transformations with positive diagonal elements. Therefore  $A'A = B'B$ . Thus  $(AB^{-1})' \cdot AB^{-1} = E$ ;  $AB^{-1} = P$  is orthogonal. Thus  $A = PB$ . Uniqueness follows from the uniqueness of the triagonal transformation that carries a quadratic form into its canonical form.

- 958  $A'A$  is a symmetric matrix with positive proper values  $\lambda_1, \dots, \lambda_n$ . Therefore

$$A'A = P^{-1} \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}P.$$

Consider the matrix

$$B = P^{-1} \text{diag}\{\mu_1, \mu_2, \dots, \mu_n\}P,$$

where  $\mu_i$  is the principal square root of  $\lambda_i$ .

Clearly  $B$  is symmetric and has positive proper values;  $B^2 = A'A$ . Therefore  $AB^{-1} = Q$  is orthogonal;  $A = QB$ .

- 959 A translation of the origin of a central surface to its center is such that if  $X$  is an arbitrary point, then after the translation either the surface contains both  $X$ ,  $-X$ , or neither. In this way the first degree terms are eliminated from the equation of the surface.

The details are as follows. The transformation needed has the form  $X = X_0 + X'$ , where  $X_0$  is the fixed translation vector. The equation of the surface becomes

$$AX' \cdot X' + 2(AX_0 + B)X' + AX_0 \cdot X_0 + 2BX_0 + C = 0.$$

A necessary and sufficient condition that a center exist is that the equation  $AX_0 + B = 0$  have a solution  $X_0$ ; and a necessary and sufficient condition for this is that the rank of the matrix  $A$  be equal to the rank of the matrix  $(A, B)$ .

- 960 A preliminary translation of the origin to the center of the surface transforms the equation into

the form  $AX \cdot X + \gamma = 0$ .

If  $r$  is the rank of  $A$  and  $\alpha_1, \dots, \alpha_r$  are the non-zero proper values of  $A$ , then a further orthogonal transformation will transform the equation of the surface into the canonical form  $\alpha_1 x_1^2 + \dots + \alpha_r x_r^2 + \gamma = 0$ .

- 961 The surface will not have a center if the rank of  $(A, B)$  exceeds the rank of  $A$ . This can only occur if  $R = \text{rank } A < n$ . Let  $R$  be the entire space; let  $P$  be the space  $AR$ , let  $Q$  be the orthogonal complement of  $P$ . Then if  $Y$  is an arbitrary vector in  $Q$ , the relation  $AY = 0$  must hold. Thus  $|AY|^2 = AY \cdot AY = Y \cdot AAY = 0$ , since  $AA Y \in P$ . Let  $B$  have the decomposition  $B = B_1 + B_2$ ,  $B_1 \in P$ ,  $B_2 \in Q$ . Then  $B_2 \neq 0$ ; otherwise  $B$  would lie in  $P$ , and  $\text{rank } (A, B)$  would be  $r$ . Set  $B_1 = AX_0$ . In other words let  $X_0$  be the inverse image of  $B_1$ . The translation  $X = X_0 + X'$  transforms the equation of the surface into

$$AX' \cdot X' + 2B_2 X' + c' = 0.$$

We now make the further translation  $X' = aB_2 + X''$ . Then  $AX' \cdot X' = a^2 AB_2 \cdot B_2 + 2aAB_2 \cdot X'' + AX'' \cdot X'' = AX'' \cdot X''$ . But  $AB_2 = 0$ ; thus the equation is finally transformed into

$$AX'' \cdot X'' + 2B_2 X'' + 2a |B_2|^2 + c' = 0.$$

If we choose  $a = -\frac{1}{2} c' / |B_2|^2$ , the equation becomes

$$AX'' \cdot X'' + 2B_2 X'' = 0.$$



Now we make an orthogonal coordinate transformation to a basis of orthogonal unit proper vectors of the matrix  $A$ , taking one of these vectors to be the unit vector collinear with  $B_2$  but oppositely directed. This is possible since  $B_2$  is a proper vector of  $A$ . In terms of the new basis, the equation has the form

$$\lambda_1 x_1^2 + \dots + \lambda_r x_r^2 - 2\beta_2 x_{r+1} = 0,$$

where  $\beta_2 = |B_2|$ . Finally we divide by  $\beta_2$ .

962 The matrix  $A$  induces a linear transformation of the space into the subspace spanned by the vectors  $A_1, A_2, \dots, A_n$ , the latter being the column vectors of  $A$ . The assertion follows directly from this.

963 Let  $e_1, e_2, \dots, e_q$  be a basis of  $Q$ ; then  $Q'$  is the space spanned by  $e'_1, e'_2, \dots, e'_q$ , where  $e'_1, e'_2, \dots, e'_q$  are the images of  $e_1, e_2, \dots, e_q$  under a certain linear transformation. Therefore  $q' \leq q$ . Moreover it is clear that  $q' \leq r$  since  $Q'$  is contained in  $R'$ . Now let  $P$  be the space complementary to  $Q$ ; the dimension of  $P$  is  $p = n - q$ . Let  $P'$  be the image of  $P$  under the linear transformation above. The dimension of the latter,  $p'$ , will not exceed  $n - q$ . But  $P' + Q' = R'$ ; therefore  $p' + q' \geq r$ . Therefore  $q' \geq r - p' \geq r + q - n$ .

964 Let  $A$  have rank  $r_1$ ;  $B$  have rank  $r_2$ ;  $BR = Q$ . The dimension of  $Q$  is  $r_2$ . Then the rank  $\rho$  of

- AB is equal to the dimension of  $ABR = AQ$ . The preceding problem shows that  $r_1 + r_2 - n \leq \rho \leq \min(r_1, r_2)$ .
- 965 Any projection operator is idempotent; i.e.  $A^2 = A$ . To see this, suppose that  $A$  projects the entire space  $R$  on the subspace  $P$ ; then a repetition of this transformation does not decrease  $P$ . Therefore  $A^2 = A$ . Conversely suppose  $A^2 = A$ . Let  $P$  be the space of all vectors of the form  $Z = AX$ ; let  $Q$  be the space of all vectors  $Y$  such that  $AY = 0$ . It is clear that  $P, Q$  are linear spaces. Their intersection is null since the two relations  $AX = Y, AY = 0$  lead to  $AX = A^2X = AY = 0$ . Thus if  $X$  is any vector,  $X = AX + (E - A)X$ . On the one hand,  $(E - A)X \in Q$ , so that  $A(E - A)X = (A - A^2)X = 0$ . On the other hand  $X$  can be written as the sum of an element in  $Q$  and an element in  $P$  (see above). Thus  $P + Q$  is the entire space;  $P, Q$  are complementary subspaces. The operator  $P$  carries the arbitrary vector  $X$  into its  $P$ -component. Thus  $A$  is a projection onto  $P$  parallel to  $Q$ .
- 966 Let  $P, Q$  be orthogonal. We combine orthonormal bases of  $P, Q$  to form an orthonormal basis of the entire space. In terms of this basis the matrix of  $A$  has the diagonal form
- $$\tilde{A} = \text{diag}\{1, 1, \dots, 1, 0, 0, \dots, 0\},$$
- where all the ones or all the zeros may be missing.
- If any other orthonormal basis is chosen, the projection will have the matrix  $A = B^{-1}\tilde{A}B$ , where

$B$  is some orthogonal matrix. It is clear that  $A$  is symmetric.

Conversely, let  $A$  be an arbitrary symmetric matrix, with  $A^2 = A$ . The spaces  $P = AR$ ,  $Q = (E - A)R$  are clearly orthogonal; and the relations  $AX \cdot (E - A)Y = X \cdot A'(E - A)Y = X \cdot (A - A^2)Y = 0$  hold.

967 Let  $A$  be a skew-symmetric matrix. We first notice that if  $X$  is an arbitrary vector, the relation  $AX \cdot X = 0$  holds. Indeed we have

$$AX \cdot X = X \cdot A'X = X \cdot (-AX) = -AX \cdot X.$$

Let  $\lambda = \alpha + \beta i$  be a proper value of the matrix  $A$ ; let  $U = X + Yi$  be the corresponding proper vector. Then

$$AX = X\alpha - Y\beta, \quad AY = X\beta - Y\alpha.$$

From this we conclude that  $\alpha(|X|^2 + |Y|^2) =$

$AX \cdot X + AY \cdot Y = 0$ ;  $\alpha = 0$ . Further,

$\beta X \cdot Y + \alpha |Y|^2 = AY \cdot Y = 0$  ; thus  $X \cdot Y = 0$  if

$\beta \neq 0$ . Finally we see that  $|X| = |Y|$  from the relations  $\beta(|X|^2 - |Y|^2) = AY \cdot X + AX \cdot Y = AY \cdot X - X \cdot AY = 0$ .

968 Let the skew-symmetric matrix  $A$  have the form

$$A = \begin{pmatrix} 0 & a_{12} & \dots & a_{1n} \\ -a_{12} & 0 & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ -a_{1n} & -a_{2n} & \dots & 0 \end{pmatrix}$$

We first remark that if every proper value of  $A$  is 0, then  $A = 0$ . To see this note that the sum of the products of the proper values two by two is the sum of the second order principal minors

$\sum_{i < k} a_{ik}^2$ . This can be 0 only if every element  $a_{ik}$  is 0.

Now let  $\lambda_1 = a_1 i$  be an arbitrary non-zero proper value of  $A$ . Let  $X, Y$  be the real and imaginary parts of the corresponding proper vector. By problem 967 these have the same length. Thus we can norm them simultaneously by dividing the vector  $X + Yi$  by an appropriate scalar factor. We then have the relations

$$AX = -Ya_1; \quad AY = Xa_1.$$

Using the Gram-Schmidt orthogonalization process we can construct an orthogonal matrix  $B$  having  $X, Y$  for its first two columns. Then the matrix  $P^{-1}AP$  has the form

$$P^{-1}AP = \begin{pmatrix} 0 & a_1 & \dots \\ -a_1 & 0 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \ddots \\ 0 & 0 & \dots \end{pmatrix}.$$

But this matrix must be skew-symmetric so that the elements in the first two rows not expressly written must all be zero. An inductive argument similar to that used to prove Schur's

theorem (see the remarks after problem 931) shows that every skew-symmetric matrix can be transformed into the form

$$\text{diag}\left\{\begin{bmatrix} 0 & a_1 \\ -a_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & a_2 \\ -a_2 & 0 \end{bmatrix}, \dots\right\}$$

by an orthogonal transformation.

969 Suppose

$$B = (E - A)(E + A)^{-1}$$

Then

$$B' = \{(E - A)^{-1}\}' (E - A)' = (E - A)^{-1}(E + A) = B^{-1}.$$

Thus

$$B + E = (E - A)(E + A)^{-1} + (E + A)(E + A)^{-1} = 2(E + A)^{-1},$$

and therefore

$$\det[B + E] \neq 0.$$

Conversely, suppose

$$B' = B^{-1}, \det[B + E] \neq 0.$$

Then we can take the matrix  $(E + B)^{-1}(E - B)$  as a possible matrix  $A$ . It is easy to see that  $A$  is skew-symmetric.

970 Let  $A$  be an arbitrary orthogonal matrix. Then if  $X, Y$  are arbitrary vectors, we have

$$AX \cdot AY = X \cdot A'AY = X \cdot Y$$

Let

$$\lambda = \alpha + \beta i$$

be a proper value of the matrix  $A$ ; let

$$U = X + Yi$$

be the corresponding proper vector. Then

$$AX = X\alpha - Y\beta, \quad AY = Y\alpha + X\beta.$$

Thus

$$|X|^2 = X \cdot X = AX \cdot AX = \alpha^2 |X|^2 + \beta^2 |Y|^2 - (X2\alpha\beta) \cdot Y,$$

$$|Y|^2 = Y \cdot Y = AY \cdot AY = \alpha^2 |Y|^2 + \beta^2 |X|^2 + (X2\alpha\beta) \cdot Y.$$

We obtain the relation  $\alpha^2 + \beta^2 = 1$

by adding the last two equations.

- 971 If  $\beta \neq 0$ , we can subtract the two equations above and obtain the relation

$$\beta(|X|^2 - |Y|^2) + (X2\alpha) \cdot Y = 0.$$

On the other hand

$$X \cdot Y = AX \cdot AY = (\alpha^2 - \beta^2)X \cdot Y + \alpha\beta(|X|^2 - |Y|^2).$$

Thus

$$\alpha(|X|^2 - |Y|^2) - 2\beta X \cdot Y = 0.$$

And finally,

$$X \cdot Y = |X|^2 - |Y|^2 = 0.$$

- 972 1. Let  $\lambda = \alpha + \beta i = \cos \varphi + i \sin \varphi$  be a non-real proper value of the matrix. We construct an orthogonal matrix  $Q$  having for its first two columns the real and imaginary parts of the proper vector that corresponds to  $\lambda$ . Then

$$Q^{-1}AQ = \begin{pmatrix} \cos \varphi & \sin \varphi & \dots \\ -\sin \varphi & \cos \varphi & \dots \\ 0 & 0 & \dots \\ 0 & 0 & \dots \\ \cdot & \cdot & \cdot \\ 0 & 0 & \dots \end{pmatrix}.$$

Since the matrix  $Q^{-1}AQ$  is orthogonal, the sum of the squares of the elements of each row is 1. Therefore all the elements in the first two rows not explicitly written down must be 0.

2. Let  $\lambda = \pm 1$  be a real characteristic number of  $A$ ; let  $X$  be the normalized proper vector corresponding to  $\lambda$ .

We construct an orthogonal matrix  $Q$ , the first column of which is the vector  $X$ . Then

$$Q^{-1}AQ = \begin{pmatrix} \pm 1 & \dots \\ 0 & \dots \\ \dots & \dots \\ 0 & \dots \end{pmatrix}.$$

Using the same argument as before we see that all the elements in the first row, except the first, are 0. The proof is completed by induction (see the argument following problem 931).

973

a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix};$	f) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix};$	k) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix};$
b) $\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$	g) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix};$	l) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix};$
c) $\begin{pmatrix} -3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix};$	h) $\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix};$	m) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix};$
d) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix};$	i) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix};$	n) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$

$$e) \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}; \quad j) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \quad o) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

974

$$a) \begin{pmatrix} 1 & 1 & & \\ 0 & 1 & & \\ & & 1 & 1 \\ & & 0 & 1 \end{pmatrix}; \quad b) \begin{pmatrix} 1 & 1 & & \\ 0 & 1 & 1 & \\ & 0 & 1 & 1 \\ & & 0 & 1 \end{pmatrix};$$

$$c) \begin{pmatrix} \varepsilon & & & \\ & \varepsilon^2 & & \\ & & \ddots & \\ & & & \varepsilon^{n-1} \end{pmatrix},$$

where  $\varepsilon = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$ .

975 The box

$$\begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda & 1 \end{pmatrix}$$

of dimension  $> 1$  cannot have finite order.

Remark. The result of this problem is not valid over a field of nonzero characteristic.

For example, if the characteristic of the field is 2, the relation  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 = E$  holds.

976 Let  $A$  be the given matrix and  $B = C^{-1}AC$  be its Jordan canonical form. The canonical matrix  $B$



is triangular and its diagonal elements are the proper values of the matrix  $A$ ; each one appears on the diagonal as many times as its multiplicity in the characteristic polynomial. Thus

$B'_m = (C'_m)^{-1} A'_m C'_m$  (see problem 531). Therefore the matrices  $A'_m$ ,  $B'_m$  have the same characteristic polynomials. If the rows and columns of  $B'_m$  are properly numbered it has triangular form and therefore its proper values are its diagonal elements; these are obviously the products of the proper values of  $A$  taken  $m$  at a time.

977 It is obvious that the matrices  $A - \lambda E$ ,  $A' - \lambda E$  have the same elementary divisors. Therefore  $A$ ,  $A'$  can be transformed into the same canonical form and therefore one can be transformed into the other.

978 Set  $A = CD$ , where  $C$ ,  $D$  are symmetric matrices and  $C$  is nonsingular. Then  $A' = DC$ ; therefore  $A' = C^{-1}AC$ . Therefore the matrix  $C$  is a matrix that transforms  $A$  into  $A'$ .

Set  $A = SBS^{-1}$ , where  $B$  is a canonical matrix:

$$B = \begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_k \end{pmatrix}, \text{ where } B_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \ddots & \ddots \\ & & & 1 & \\ & & & & \lambda_i \end{pmatrix}.$$

Then  $A' = S'^{-1}B'S'$ . Let  $H_i$  be the matrix

$$H_i = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdot & & \\ & & & \cdot & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}.$$

It is clear that  $B'_i = H_i^{-1}B_iH_i$ . Therefore

$$B' = H^{-1}BH, \text{ where } H = \text{diag}\{H_1, H_2, \dots, H_k\}.$$

Thus  $A' = S'^{-1}H^{-1}BHS' = S'^{-1}H^{-1}S^{-1}ASHS' = C^{-1}AC$ , where  $C = SHS'$ . The matrix  $C$  is obviously symmetric. Set  $D = C^{-1}A$ , then  $D' = A'C^{-1} = C^{-1}ACC^{-1} = D$ . Thus the matrix  $D$  is also symmetric;  $A = CD$ .

979 Let  $\det[A - \lambda E] = (-1)^n(\lambda^n - c_1\lambda^{n-1} - c_2\lambda^{n-2} - \dots - c_n)$ . First note that  $p_1 = \text{Sp } A = c_1$ . As an induction hypothesis suppose that  $p_1 = c_1, p_2 = c_2, \dots, p_{k-1} = c_{k-1}$ , and let us establish the relation  $p_k = c_k$ . By construction  $A_k = A^k - p_1A^{k-1} - p_2A^{k-2} - \dots - p_{k-1}A = A^k - c_1A^{k-1} - c_2A^{k-2} - \dots - c_{k-1}A$ . Therefore

$$\begin{aligned} \text{Sp } A_k &= kp_k = \text{Sp } A^k - c_1\text{Sp } A^{k-1} - \dots - c_{k-1}\text{Sp } A \\ &= S_k - c_1S_{k-1} - \dots - c_{k-1}S_1, \end{aligned}$$

where  $S_1, S_2, \dots, S_k$  are power sums of the proper values of the matrix  $A$ . But by Newton's formula

$$S_k - c_1S_{k-1} - \dots - c_{k-1}S_1 = kc_k. \quad \text{Therefore } p_k = c_k.$$

Now by the Hamilton-Cayley theorem,

$$B_n = A^n - c_1 A^{n-1} - \dots - c_{n-1} A - c_n E = 0 \quad . \quad \text{Finally} \\ AB_{n-1} = A_n = c_n E \quad , \text{ so that } B_{n-1} = c_n A^{-1}.$$

980 If  $C$  has the form

$$C = \begin{pmatrix} 0 & c_{12} & \dots & c_{1n} \\ c_{21} & 0 & \dots & c_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ c_{n1} & c_{n2} & \dots & 0 \end{pmatrix} , \text{ with zero diagonal,}$$

the assertion is clear. Indeed we can take

$$X = \text{diag}\{\alpha_1, \alpha_2, \dots, \alpha_n\},$$

with arbitrary distinct diagonal elements. Then we can set

$$Y = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix} , \text{ and note that}$$

$$XY - YX = \begin{pmatrix} 0 & b_{12}(\alpha_2 - \alpha_1) & \dots & b_{1n}(\alpha_n - \alpha_1) \\ b_{21}(\alpha_1 - \alpha_2) & 0 & \dots & b_{2n}(\alpha_n - \alpha_2) \\ \cdot & \cdot & \cdot & \cdot \\ b_{n1}(\alpha_1 - \alpha_n) & b_{n2}(\alpha_2 - \alpha_n) & \dots & 0 \end{pmatrix}$$

The problem is solved by taking

$$b_{ik} = c_{ik}/(\alpha_k - \alpha_i); i \neq k.$$

To finish, it is sufficient to show that every matrix  $C$  with zero trace is similar to a matrix with zero diagonal. This we prove by induction as follows. If  $\text{Sp } C = 0$ , then  $C$  cannot have the form  $C = \mu E$  unless  $\mu = 0$ . Therefore there must be a vector  $U$  such that  $U, CU$  are independent. Let  $W$  be a nonsingular matrix having  $U, CU$  for its first two columns. Then

$$W^{-1}CW = C' = \begin{pmatrix} 0 & \gamma_{12} & \gamma_{13} & \cdots & \gamma_{1n} \\ 1 & \gamma_{22} & \gamma_{23} & \cdots & \gamma_{2n} \\ . & . & . & . & . \\ 0 & \gamma_{n2} & \gamma_{n3} & \cdots & \gamma_{nn} \\ . & . & . & . & . \end{pmatrix} = \begin{bmatrix} O & P \\ Q & M \end{bmatrix}.$$

Therefore,  $\text{Sp } M = \text{Sp } C = 0$ .

By an appropriate induction hypothesis we have  $M = S^{-1}M'S$ , where  $M'$  is a matrix with zero diagonal. Thus the matrix

$$C'' = \begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix}^{-1} C' \begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix} = \begin{bmatrix} 0 & PS \\ S^{-1}Q & S^{-1}MS \end{bmatrix}$$

has zero diagonal.

- 981 The series  $H = I + CC^* + C^2 C^{*2} + \dots$  converges, since  $\|C\| < 1$  (see problem 956). Direct substitution shows that  $H$  satisfies the relation given.

See P. Stein, J. Research Nat. Bur. Stds.  
48 (1952) 82-83.

- 982 Similar to 981;  $\|C^k BC^{*k}\| \leq \|B\| \cdot \|C\|^{2k}$ .
- 983 Set  $K = I - A$ ,  $L = I + A$ ; note that  $K^{-1}$  exists. Define  $C$  by  $C = LK^{-1} = K^{-1}L = (I + A)(I - A)^{-1} = (I - A)^{-1}(I + A)$ . By problem 931, all proper values of  $C$  lie in the unit circle, since the quantity  $(1 + \rho)(1 - \rho)^{-1} = (1 + x + iy)(1 - x - iy)$  has modulus less than 1 if  $x$  is a negative number.

We are to assume that there is a positive-definite hermitian matrix  $G$  such that

$$(2) \quad G - CGC^* = K^{-1}(2R)(K^{-1})^*,$$

the right member of the latter equation being positive-definite hermitian, since  $R$  is.

We write (2) in the form

$$G - K^{-1}LGL^* K^{*-1} = K^{-1}(2R) K^{*-1}, \text{ or}$$

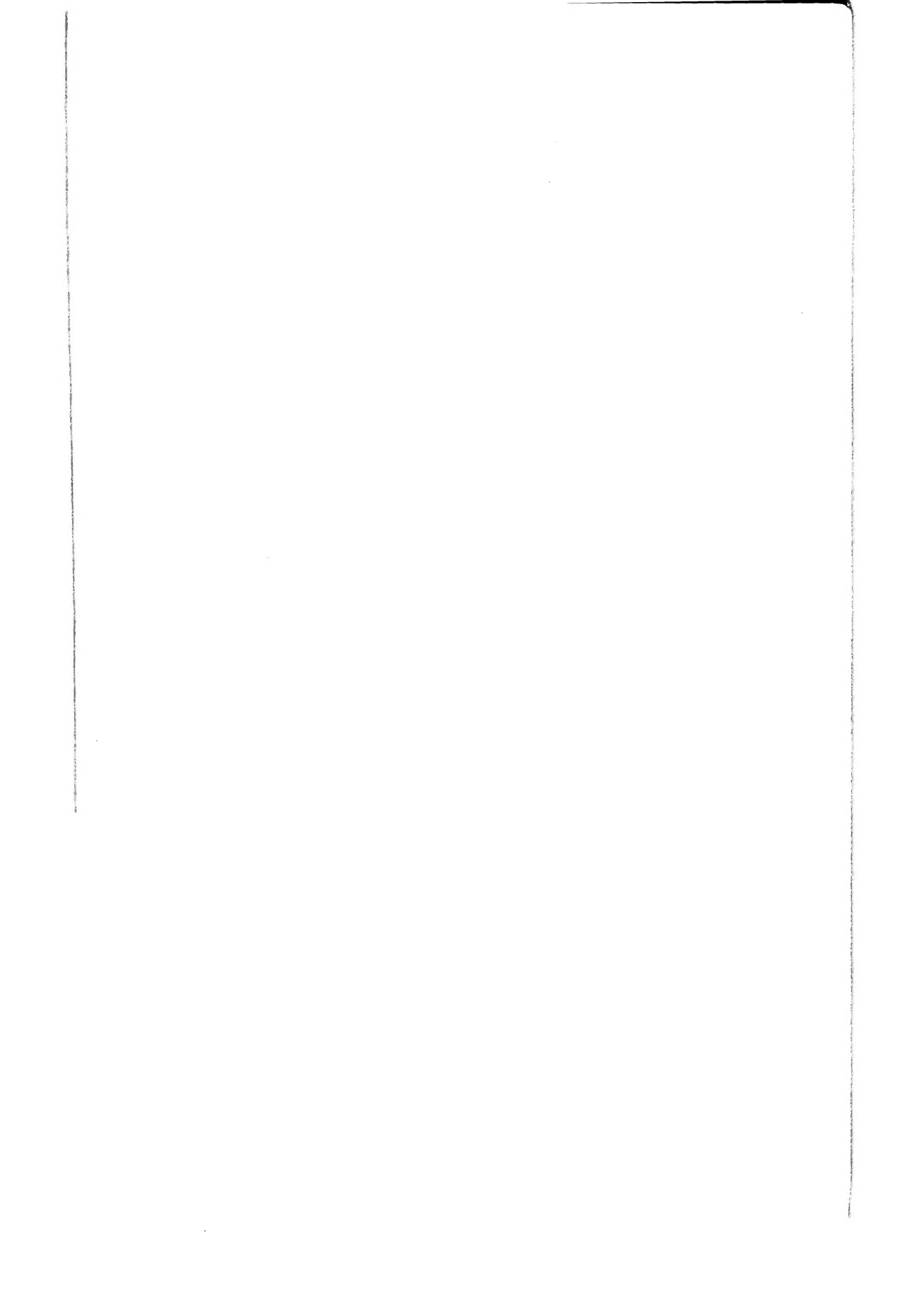
$$KGK^* - LGL^* = 2R.$$

Direct substitution yields

$$\begin{aligned} KGK^* &= (I - A)G(I - A^*) = G - GA^* - AG + AGA^*, \\ -LGL^* &= -(I + A)G(I + A^*) = -G - GA^* - AG - AGA^*, \end{aligned}$$

which establishes the assertion.

See O. Taussky, SIAM Journal 9 (1961), 640-643, and many later papers.





Comprising approximately 1,000 problems in higher algebra, with hints and solutions, this book is recommended as an adjunct text, as a problem book, and for self study. The following is a sampling of the variety of problems in this collection:

1. Calculation of determinants. Inductive methods. Partitioning. Cases in which the entries are large numbers.
2. Systems of linear equations. Explanation of the point of view that the set of solutions forms a vector space. Use of modern terminology.
3. Matrices. Inequalities that involve rectangular matrices.
4. Polynomials and rational functions. Classical theorems on bounds for the roots with complete proofs.
5. Symmetric functions. Application to polynomial equations.
6. Linear algebra. Elementary and moderately advanced theorems. Euclidean norms. Jordan canonical forms. Conditions that the proper values of a matrix lie in the left half plane.

A feature of the book is that every problem, or at the most every group of two or three problems, is independent, and sections of the book can be sampled or omitted at the user's discretion.

**W. H. FREEMAN AND COMPANY**

660 Market Street, San Francisco, California 94104

Warner House, 48 Upper Thames Street, London, E. C. 4

Freeman